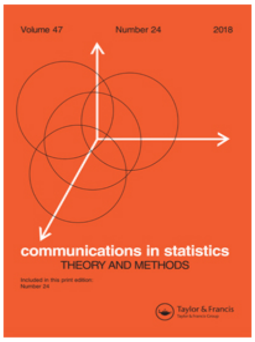

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Some Hermite–Hadamard type integral inequalities for multidimensional general preinvex stochastic processes

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ABSTRACT
In this study, we initially defined general preinvexity for real valued stochastic processes. Correspondingly, we also established multidimensional general preinvex stochastic processes which are a significant class of stochastic processes for optimization. Moreover, we obtained Hermite–Hadamard type integral inequalities for before mentioned processes using mean-square integrability.

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1. Introduction and preliminaries

It has become famous the following HHII for convex functions (Hadamard 1893):

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

HHII for convex functions has received renewed attention in recent years and the remarkable varieties of refinements and generalizations can be found in the literature. For example, Dragomir (2001) proved HHII for convex functions on the coordinates in a rectangle from the plane.

In this context, Youness (1999) defined E-convexity such that a function $f:\mathbb{R}^n \to \mathbb{R}$ is said to be E-convex on a set $M \subset \mathbb{R}^n$ iff there is a map $E:\mathbb{R}^n \to \mathbb{R}^n$ such that M is an E-convex set and

$$f\left(\lambda E(t) + (1-\lambda)E(s)\right) \le \lambda f(E(t)) + (1-\lambda)f(E(s))$$

for each $t,\ s \in M$, and $\lambda \in [0,1]$. Also, Cristescu (2004) derived the following HHII for another type of convexity, called φ-convexity:

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$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \leq \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x)dx \leq \frac{f(\varphi(a))+f(\varphi(b))}{2}$$

It is necessary to know that preinvexity indicates a generalization of convexity for functions. Therefore, HHII for preinvex functions were obtained by many researchers (Hanson 1981; Ben-Israel and Mond 1986; Pini 1991; Mohan and Neogy 1995; Weir and Mond 1988; Yang and Li 2001; Noor 2005, 2007; Mishra and Giorgi 2008). Let us recall some known results concerning invexity and preinvexity for functions:

A set $I \subseteq \mathbb{R}^n$ is called invex with respect to the continuous function $\eta : I \times I \to \mathbb{R}^n$, if $x + \lambda\eta(y, x) \in I$ for all $x, y \in I,\ \ \lambda \in [0, 1]$ (Mohan and Neogy 1995).

Mohan and Neogy (1995) proved that an invex function is also preinvex under following Condition C:

Condition C. Let $I \subseteq \mathbb{R}^n$ be invex with respect to $\eta : I \times I \to \mathbb{R}^n$. It is told that the function η satisfies Condition C, if

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y);\ \ \eta(x, y + \lambda\eta(x, y)) = (1-\lambda)\eta(x, y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Let $I \subseteq \mathbb{R}^n$ be invex with respect to $\eta : I \times I \to \mathbb{R}^n$, the function $f : I \to \mathbb{R}$ is called preinvex with respect to η, if

$$f(x + \lambda\eta(y, x)) \leq \lambda f(y) + (1-\lambda)f(x)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$. If η satisfies Condition C, then (Noor 2005):

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_{a}^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(b)}{2}$$

Matloka (2013) obtained HHII for (h_1, h_2)-preinvex functions on the coordinates in a rectangle from the plane such that $h_1, h_2 \not\equiv 0$ and f is a non negative function. Moreover, Set, Sarıkaya, and Akdemir (2014) showed HHII for φ-convex functions on the coordinates in a rectangle from the plane. De la Cal and Carcamo (2006) obtained multidimensional HHII in 2006. Nowadays, Ellahi, Farid, and Rehman (2015) derived HHII for s-convex functions on n-coordinates. Viloria and Cortez (2018) verified HHII for harmonically convex functions on n-coordinates. Awan et al. (2017) defined general preinvexity (namely φ-preinvexity) for functions and obtained the following HHII for these functions:

$$f\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right) \leq \frac{1}{\eta(\varphi(b),\varphi(a))} \int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} f(\varphi(x))d\varphi(x)$$
$$\leq \frac{f(\varphi(a))+f(\varphi(a)+\eta(\varphi(b),\varphi(a)))}{2} \leq \frac{f(\varphi(a))+f(\varphi(b))}{2}$$

In probabilistic words, it can be said that

$$f(EX) \leq_{cx} Ef(X) \leq_{cx} Ef(X^*),\ \ f \in C_{cx}$$

where E denotes mathematical expectation, X (respectively, X^*) is a random variable having the uniform distribution on the interval $\left[\varphi(a), \varphi(a)+\eta\left(\varphi(b), \varphi(a)\right)\right]$ (respectively, on the set $\left\{\varphi(a), \varphi(a)+\eta\left(\varphi(b), \varphi(a)\right)\right\}$ or $\left\{\varphi(a), \varphi(b)\right\}$), and it should be observed that $EX = EX^* = \frac{2\varphi(a)+\eta\left(\varphi(b), \varphi(a)\right)}{2}$, and $\mathcal{C}_{cx}$ is the set of all real convex functions on $\left[\varphi(a), \varphi(a)+\eta\left(\varphi(b), \varphi(a)\right)\right]$; $\leq_{cx}$ stands for the so called convex order of random variables (De la Cal and Carcamo 2006).

Concordantly, there are satisfactory evidence on similar results belong to stochastic processes. Nikodem (1980) proposed CSP and its some properties. Some applications to stochastic convexity verified by Shaked and Shanthikumar (1988). Jensen-convex, λ-convex stochastic processes were introduced by Skowronski (1992). The classical HHII to CSP was extended using mean-square integrability by Kotrys (2012). Moreover Akdemir, Bekar, and İşcan (2014) defined $P_\eta SP$. Hereunder, the process $X : I \subset \mathbb{R} \times \Omega \to \mathbb{R}$ is known preinvex with respect to η, iff

$$X((t+\lambda\eta(s,t)), \ \cdot) \leq (1-\lambda)X(t, \ \cdot) + \lambda X(s, \ \cdot)$$

for all $t, s \in I$ and $\lambda \in [0,1]$. If η satisfies Condition C (Mohan and Neogy 1995), then the following HHII for $X : [u, u+\eta(v,u)] \times \Omega \to \mathbb{R}_+$ holds almost everywhere:

$$X\left(\frac{2u+\eta(v,u)}{2}, \cdot\right) \leq \frac{1}{\eta(v,u)} \int_u^{u+\eta(v,u)} X(t,\cdot)dt \leq \frac{X(u,\cdot)+X(v,\cdot)}{2}$$

Recently, Aliyev, Khaniyev, and Bekar (2009), Aliyev and Khaniyev (2014), and Aliyev (2017) verified important results on some class of stochastic processes. Sarikaya, Kiriş, and Çelik (2016) defined the φ_h-CSP and presented HHII for these processes. Karahan and Okur (2018) investigated MCSP and obtained HHII for these processes. According to this study, the n-dimensional interval is defined as $\Lambda^n = \prod_{i=1}^n [u_i, v_i]$, where u_i, v_i are real numbers such that $u_i < v_i$ for $i = 1, 2, ..., n;\ n \geq 2$. Then, a stochastic process $X : \Lambda^n \times \ \Omega \to \mathbb{R}$ is said to be multidimensional convex on Λ^n if the following inequality holds:

$$X((\lambda \boldsymbol{t} + (1-\lambda)\boldsymbol{s}), \cdot) \leq \lambda X(\boldsymbol{t}, \cdot) + (1-\lambda)X(\boldsymbol{s}, \cdot), \ (a.e.)$$

for all $\boldsymbol{t} = (t_1, t_2, ..., t_n)$, $\boldsymbol{s} = (s_1, s_2, ..., s_n) \in \Lambda^n$ and $\lambda \in [0,1]$. Nowadays, Okur and Karahan (2019) derived some important results for s-convexity of MSP.

In the light of these, we principally defined general preinvexity for multidimensional processes and obtained Hermite–Hadamard type inequalities for these processes.

2. Main results

This section contains two sub-sections. In the first sub-section, we initially defined general preinvexity for real valued stochastic processes, immediately afterwards we derived the Hermite–Hadamard inequality for these processes. In the second sub-section, we defined multidimensional general preinvex stochastic process and proved the Hermite–Hadamard type inequality for these processes. Throughout this article, the expression “almost everywhere” is shown with the symbol “(a.e.)” shortly.

2.1. General preinvex stochastic processes (G$P_\eta\varphi$SP)

In this sub-section, we define general preinvex stochastic processes on real number line. For this reason, we assume that $I \subseteq \mathbb{R}$ be a non empty closed set and $\eta : I \times I \to \mathbb{R}$, $\varphi : I \to I$ be arbitrary functions.

Definition 2.1. *A set $I \subseteq \mathbb{R}$ is called a general invex set with respect to η and φ, iff*

$$\varphi(t) + \lambda\eta\big(\varphi(s), \varphi(t)\big) \in I, \quad \forall t, s \in \mathbb{R} : \varphi(t), \quad \varphi(s) \in I, \quad \lambda \in [0, 1]$$

Condition C for $\mathbf{GP_{\eta\varphi}SP}$. Let $\eta : I \times I \to \mathbb{R}$ hold the following criterions

$$\eta\big(\varphi(t), \varphi(t) + \lambda\eta\big(\varphi(s), \varphi(t)\big)\big) = -\lambda\eta\big(\varphi(s), \varphi(t)\big);$$
$$\eta\big(\varphi(s), \varphi(t) + \lambda\eta\big(\varphi(s), \varphi(t)\big)\big) = (1 - \lambda)\eta\big(\varphi(s), \varphi(t)\big);$$
$$\forall t, s \in \mathbb{R} : \varphi(t), \varphi(s) \in I, \quad \lambda \in [0, 1]$$

Definition 2.2. *Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process on the general invex set I with respect to η and φ. Then, the stochastic process X is called $GP_{\eta\varphi}SP$ with respect to η and φ iff*

$$X\big(\big(\varphi(t) + \lambda\eta\big(\varphi(s), \varphi(t)\big)\big), \ \cdot\big) \le \lambda X(\varphi(s), \ \cdot) + (1 - \lambda)X(\varphi(t), \ \cdot)$$
$$\forall t, s \in \mathbb{R} : \varphi(t), \varphi(s) \in I, \ \lambda \in [0, 1]$$

Now, we can give the following lemma as in Kotrys (2012).

Lemma 2.1 . *Let $(\Omega, \Im, P)$ be an arbitrary probability space and $X : I \times \Omega \to \mathbb{R}$ be a stochastic process on the general invex set I with respect to η and φ. If φ is a continuous function on I, then X can be known as*

(i) *mean-square continuous (differentiable) on I,*
(ii) *monotonic if it is increasing or decreasing,*
(iii) *mean-square differentiable at a point $\varphi(t) \in I$,*
(iv) *mean-square integrable on $[\varphi(u), \varphi(u) + \eta(\varphi(v), \varphi(u))] \subseteq I$.*

Proof. It is easily obtained by means of Definition 2.1.

Theorem 2.1. *Under the assumptions of Lemma 2.1, let $X : [\varphi(u), \varphi(u) + \eta(\varphi(v), \varphi(u))] \times \Omega \to \mathbb{R}_+$ be a $GP_{\eta\varphi}SP$ with respect to η and φ. If η fulfils the criterions of Condition C, then the following inequalities holds almost everywhere*

$$X\left(\frac{2\varphi(u) + \eta(\varphi(v), \varphi(u))}{2}, \cdot\right)$$
$$\le \frac{1}{\eta(\varphi(v), \varphi(u))} \int_{\varphi(u)}^{\varphi(u)+\eta(\varphi(v), \varphi(u))} X(\varphi(t), \cdot) d\varphi(t) \le \frac{X(\varphi(u), \cdot) + X(\varphi(v), \cdot)}{2}$$

Proof. By means of Condition C, we get

$$X\left(\frac{2\varphi(u) + \eta(\varphi(v), \varphi(u))}{2}, \cdot\right)$$
$$\le \frac{1}{2}\big[X(\varphi(u) + \lambda\eta(\varphi(v), \varphi(u)), \ \cdot) + X\big(\varphi(u) + (1 - \lambda)\eta(\varphi(v), \varphi(u)), \ \cdot\big)\big]$$

Thus, integrating the above inequality on $[0,1]$

$$X\left(\frac{2\varphi(u)+\eta(\varphi(v),\varphi(u))}{2},\cdot\right) \le \frac{1}{\eta(\varphi(v),\varphi(u))}\int_{\varphi(u)}^{\varphi(u)+\eta(\varphi(v),\varphi(u))} X(\varphi(t),\cdot)d\varphi(t)$$

Integrating the inequality in Definition 2.2 on $[0,1]$

$$\frac{1}{\eta(\varphi(v),\varphi(u))}\int_{\varphi(u)}^{\varphi(u)+\eta(\varphi(v),\varphi(u))} X(\varphi(t),\cdot)d\varphi(t) \le \frac{X(\varphi(u),\cdot)+X(\varphi(v),\cdot)}{2}$$

This finishes the proof of Theorem 2.1.

2.2. Multidimensional general preinvex stochastic processes ($MGP_\eta\varphi$ SP)

In this sub-section, we give definition of multidimensional general invexity and preinvexity for stochastic processes with respect to η and φ. Hereinafter, note that u_i, v_i are real numbers such that $u_i < v_i$ for $i = 1,2,\dots,n,\ \ n \ge 2$,

$$\varphi(\boldsymbol{t}) := \big(\varphi(t_1),\dots,\varphi(t_n)\big) \equiv \big(\textstyle\bigwedge_{i=1}^{n}\varphi(t_i)\big) \in I_{\eta\varphi},$$
$$\varphi(\boldsymbol{s}) := \big(\varphi(s_1),\dots,\varphi(s_n)\big) \equiv \big(\textstyle\bigwedge_{i=1}^{n}\varphi(s_i)\big) \in I_{\eta\varphi}$$

where $I_{\eta\varphi} \subseteq \mathbb{R}^n$ be a non empty closed set. Also, assume that $\eta : I_{\eta\varphi} \times I_{\eta\varphi} \longrightarrow \mathbb{R}^n$ and $\varphi : I_{\eta\varphi} \longrightarrow I_{\eta\varphi}$ be arbitrary functions.

Definition 2.3. A set $I_{\eta\varphi} \subseteq \mathbb{R}^n$ is called a multidimensional general invex set with respect to η and φ, iff

$$\varphi(\boldsymbol{t}) + \lambda\eta\big(\varphi(\boldsymbol{s}),\varphi(\boldsymbol{t})\big) \in I_{\eta\varphi}$$

for all $\varphi(\boldsymbol{t}) := \big(\varphi(t_1),\dots,\varphi(t_n)\big),\ \ \varphi(\boldsymbol{s}) := \big(\varphi(s_1),\dots,\varphi(s_n)\big) \in I_{\eta\varphi}$ and $\lambda \in [0,1]$.

Note that, the function $\eta : I_{\eta\varphi} \times I_{\eta\varphi} \longrightarrow \mathbb{R}^n$ holds criterions of Condition C for all $\varphi(\boldsymbol{t}), \varphi(\boldsymbol{s}) \in I_{\eta\varphi}$ (Awan et al. 2017).

Definition 2.4. A stochastic process $X : I_{\eta\varphi} \times \Omega \to \mathbb{R}$ is said to be $\mathrm{MGP}_{\eta\varphi}\mathrm{SP}$ with respect to η and φ on $I_{\eta\varphi}$ if the following inequality holds almost everywhere

$$X\Big(\big(\varphi(\boldsymbol{t}) + \lambda\eta\big(\varphi(\boldsymbol{s}),\varphi(\boldsymbol{t})\big)\big),\cdot\Big) \le \lambda X(\varphi(\boldsymbol{s}),\cdot) + (1-\lambda)X(\varphi(\boldsymbol{t}),\cdot)$$

for all $\varphi(\boldsymbol{t}),\ \ \varphi(\boldsymbol{s}) \in I_{\eta\varphi}$ and $\lambda \in [0,1]$. If the above inequality is reversed then X is said to be general pre-concave with respect to η and φ on $I_{\eta\varphi}$.

Additionally,

(i) if $\eta\big(\varphi(\boldsymbol{s}),\varphi(\boldsymbol{t})\big) = \eta\big(\varphi(s),\varphi(t)\big)$ for all $\varphi(t),\varphi(s) \in I$, then $\mathrm{GP}_{\eta\varphi}$ SP
(ii) if $\eta\big(\varphi(\boldsymbol{s}),\varphi(\boldsymbol{t})\big) = \eta(\boldsymbol{s},\boldsymbol{t})$ for all $\boldsymbol{t},\boldsymbol{s} \in \mathbb{R}^n$, then MP_η SP
(iii) if $\eta\big(\varphi(\boldsymbol{s}),\varphi(\boldsymbol{t})\big) = \eta(s,t)$ for all $t,s \in \mathbb{R}$, then P_η SP
(iv) if $\eta\big(\varphi(\boldsymbol{s}),\varphi(\boldsymbol{t})\big) = \varphi(\boldsymbol{s}) - \varphi(\boldsymbol{t})$ for all $\varphi(\boldsymbol{t}),\ \ \varphi(\boldsymbol{s}) \in I_{\eta\varphi}$, then MGCSP
(v) if $\eta\big(\varphi(\boldsymbol{s}),\varphi(\boldsymbol{t})\big) = \varphi(s) - \varphi(t)$ for all $\varphi(t),\varphi(s) \in I$, then GCSP
(vi) if $\eta\big(\varphi(\boldsymbol{s}),\varphi(\boldsymbol{t})\big) = \boldsymbol{s} - \boldsymbol{t}$ for all $\boldsymbol{t},\boldsymbol{s} \in \mathbb{R}^n$, then MCSP
(vii) if $\eta\big(\varphi(\boldsymbol{s}),\varphi(\boldsymbol{t})\big) = s - t$ for all $t,s \in \mathbb{R}$, then CSP

can be obtained by Definition 2.4.

Definition 2.5. A stochastic process $X : I_{\eta\varphi} \times \Omega \to \mathbb{R}$ is called MGP$_{\eta\varphi}$SP with respect to η and φ on n-coordinates if the following stochastic mappings $X^i_{\varphi(t_n)} \; : \; I \times \Omega \to \mathbb{R}$ are GP$_{\eta\varphi}$SP with respect to η and φ on I almost everywhere for all $\varphi(t) \in I$:

$$X^i_{\varphi(t_n)}(\varphi(t), \cdot) := X\Big(\Big(\bigwedge\nolimits_{k=1}^{i-1} \varphi(t_k),\;\; \varphi(t),\; \bigwedge\nolimits_{k=i+1}^{n} \varphi(t_k)\Big), \cdot\Big)$$

Lemma 2.2. *Every MGP$_{\eta\varphi}$SP is GP$_{\eta\varphi}$SP with respect to η and φ on n-coordinates almost everywhere, but converse is not true.*

Proof. Let $X : I_{\eta\varphi} \times \Omega \to \mathbb{R}$ be a MGP$_{\eta\varphi}$ SP with respect to η *and* φ on $I_{\eta\varphi}$. Consider $X^i_{\varphi(t_n)} \; : \; I \times \Omega \to \mathbb{R}$ defined by

$$X^i_{\varphi(t_n)}(\varphi(t), \cdot) := X\Big(\Big(\bigwedge\nolimits_{k=1}^{i-1} \varphi(t_k),\;\; \varphi(t),\; \bigwedge\nolimits_{k=i+1}^{n} \varphi(t_k)\Big), \cdot\Big),\;\; (a.e.)$$

for $\varphi(t) \in I$. Now for $\varphi(t), \varphi(s) \in I$ and $\lambda \in [0, 1]$ almost everywhere

$$\begin{aligned} & X^i_{\varphi(t_n)}\big((\varphi(t) + \lambda\eta(\varphi(s), \quad \varphi(t))), \cdot\big) \\ & = X\Big(\Big(\bigwedge\nolimits_{k=1}^{i-1} \varphi(t_k), \big(\varphi(t) + \lambda\eta\big(\varphi(s), \varphi(t)\big)\big),\; \bigwedge\nolimits_{k=i+1}^{n} \varphi(t_k)\Big), \cdot\Big) \\ & \le \lambda X\Big(\Big(\bigwedge\nolimits_{k=1}^{i-1} \varphi(t_k),\;\; \varphi(s),\; \bigwedge\nolimits_{k=i+1}^{n} \varphi(t_k)\Big), \cdot\Big) \\ & \quad + (1-\lambda) X\Big(\Big(\bigwedge\nolimits_{k=1}^{i-1} \varphi(t_k),\;\; \varphi(t),\; \bigwedge\nolimits_{k=i+1}^{n} \varphi(t_k)\Big), \cdot\Big) \\ & = \lambda X^i_{\varphi(t_n)}(\varphi(s), \cdot) + (1-\lambda) X^i_{\varphi(t_n)}(\varphi(t), \cdot) \end{aligned}$$

For converse we give the following counter example:

Example 2.1. Consider a stochastic process $X : [0, 1]^n \times \Omega \to \mathbb{R}$ defined as

$$X(\varphi(\boldsymbol{t}), \cdot) := X\big((\varphi(t_1), \varphi(t_2), \ldots,\;\; \varphi(t_n)), \cdot\big) = \varphi(t_1)\varphi(t_2)\ldots\varphi(t_n)$$

$$\eta : [0, 1]^n \to [0, \infty),\;\; \eta\big(\varphi(\boldsymbol{s}), \varphi(\boldsymbol{t})\big) := \varphi(\boldsymbol{s}) - \varphi(\boldsymbol{t})$$

for all $\varphi(\boldsymbol{t}),\; \varphi(\boldsymbol{s}) \in [0, 1]^n$. It is obviously a MGP$_{\eta\varphi}$ SP with respect to η and φ on n-coordinates. But for $\varphi(\boldsymbol{t}) = (1, \ldots, 1, 0),\;\; \varphi(\boldsymbol{s}) = (0, 1, \ldots, 1)$

$$X\big(\big(\varphi(\boldsymbol{t}) + \lambda\eta\big(\varphi(\boldsymbol{s}), \varphi(\boldsymbol{t})\big)\big), \cdot\big) = X((1-\lambda, 1, \ldots, 1, \lambda), \cdot) = \lambda(1-\lambda)$$

However $\lambda X(\varphi(\boldsymbol{s}), \cdot) + (1-\lambda)X(\varphi(\boldsymbol{t}), \cdot) = \lambda.0 + (1-\lambda).0 = 0$. Therefore,

$$X\big(\big(\varphi(\boldsymbol{t}) + \lambda\eta\big(\varphi(\boldsymbol{s}), \varphi(\boldsymbol{t})\big)\big), \cdot\big) > \lambda X(\varphi(\boldsymbol{s}), \cdot) + (1-\lambda)X(\varphi(\boldsymbol{t}), \cdot)$$

for $\lambda \in [0, 1]$. That case is contradiction with general preinvexity of X. This completes proof of Lemma 2.2.

Now, we can obtain HHII for MGP$_{\eta\varphi}$ SP. From here on out, assume that

$$\mathfrak{D}^n := \prod\nolimits_{i=1}^{n} \Delta^i \subseteq \mathbb{R}^n_+ \;\text{ with }\; \Delta^i := [\varphi(u_i), \varphi(u_i) + \eta(\varphi(v_i), \varphi(u_i))]$$

$$\Delta^i_+ \;:= 2\varphi(u_i) + \eta(\varphi(v_i), \varphi(u_i));\;\; \Delta^i_- \;:= \eta(\varphi(v_i), \varphi(u_i)) \;\text{ with }\; \eta(\varphi(v_i), \varphi(u_i)) > 0$$

for each $i = 1, 2, \ldots, n,\;\; n \ge 2$.

Remark 2.1. Under the assumptions of Lemma 2.1, if $X : \mathfrak{D}^n \times \Omega \to \mathbb{R}_+$ is MG$P_{\eta\varphi}$SP with respect to η and φ on $\mathfrak{D}^n$, then $X^i_{\varphi(t_n)} \ : \ \Delta^i \times \Omega \to \mathbb{R}_+$ is general preinvex stochastic process with respect to η and φ, and mean-square integrable on Δ^i. Thus

$$\begin{aligned} X^i_{\varphi(t_n)}\left(\frac{\Delta^i_+}{2},\cdot\right) &\le \frac{1}{\Delta^i_-}\int_{\varphi(u_i)}^{\Delta^i_+ -\varphi(u_i)} X^i_{\varphi(t_n)}(\varphi(t_i),\cdot)d\varphi(t_i) \\ &\le \frac{X^i_{\varphi(t_n)}(\varphi(u_i),\cdot)+X^i_{\varphi(t_n)}(\varphi(v_i),\cdot)}{2}, \ (a.e.) \end{aligned} \tag{2.1}$$

for $\Delta^i_- > 0; \ i = 1, 2, ..., n, \ n \ge 2$.

Theorem 2.2. Let $X : \mathfrak{D}^n \times \Omega \to \mathbb{R}_+$ be MG$P_{\eta\varphi}$SP with respect to η and φ on $\mathfrak{D}^n$. If the assumptions of Lemma 2.1 satisfy, then almost everywhere

$$\begin{aligned} &\sum_{i=1}^{n-1} X\left(\left(\bigwedge\nolimits_{k=1}^{i-1}\varphi(t_k),\ \frac{\Delta^i_+}{2},\frac{\Delta^{i+1}_+}{2},\bigwedge\nolimits_{k=i+2}^{n}\varphi(t_k)\right),\cdot\right) \\ &\le \sum_{i=1}^{n-1}\frac{1}{\Delta^i_-}\int_{\varphi(u_i)}^{\Delta^i_+ -\varphi(u_i)} X^{i+1}_{\varphi(t_n)}\left(\frac{\Delta^{i+1}_+}{2},\cdot\right)d\varphi(t_i) \\ &\le \sum_{i=1}^{n-1}\frac{1}{\Delta^{i+1}_-\Delta^i_-}\int_{\varphi(u_i)}^{\Delta^i_+ -\varphi(u_i)}\int_{\varphi(u_{i+1})}^{\Delta^{i+1}_+ -\varphi(u_{i+1})} X^{i+1}_{\varphi(t_n)}(\varphi(t_{i+1}),\cdot)d\varphi(t_{i+1})d\varphi(t_i) \\ &\le \sum_{i=1}^{n-1}\frac{1}{2\Delta^i_-}\int_{\varphi(u_i)}^{\Delta^i_+ -\varphi(u_i)}\left[X^{i+1}_{\varphi(t_n)}(\varphi(u_{i+1}),\cdot)+X^{i+1}_{\varphi(t_n)}(\varphi(v_{i+1}),\cdot)\right]d\varphi(t_i) \\ &\le \frac{1}{4}\sum_{i=1}^{n-1}\left[\begin{array}{l} X\left(\left(\bigwedge_{k=1}^{i-1}\varphi(t_k),\varphi(u_i),\varphi(u_{i+1}),\bigwedge_{k=i+2}^{n}\varphi(t_k)\right),\cdot\right) \\ +X\left(\left(\bigwedge_{k=1}^{i-1}\varphi(t_k),\varphi(v_i),\varphi(u_{i+1}),\bigwedge_{k=i+2}^{n}\varphi(t_k)\right),\cdot\right) \\ +X\left(\left(\bigwedge_{k=1}^{i-1}\varphi(t_k),\varphi(u_i),\varphi(v_{i+1}),\bigwedge_{k=i+2}^{n}\varphi(t_k)\right),\cdot\right) \\ +X\left(\left(\bigwedge_{k=1}^{i-1}\varphi(t_k),\varphi(v_i),\varphi(v_{i+1}),\bigwedge_{k=i+2}^{n}\varphi(t_k)\right),\cdot\right) \end{array}\right] \end{aligned} \tag{2.2}$$

Proof. Taking into account Remark 2.1 for $X^{i+1}_{\varphi(t_n)}; \ i = 1, 2, ..., n-1$, then

$$\begin{aligned} X^{i+1}_{\varphi(t_n)}\left(\frac{\Delta^{i+1}_+}{2},\cdot\right) &\le \frac{1}{\Delta^i_-}\int_{\varphi(u_{i+1})}^{\Delta^{i+1}_+ -\varphi(u_{i+1})} X^{i+1}_{\varphi(t_n)}(\varphi(t_{i+1}),\cdot)d\varphi(t_{i+1}) \\ &\le \frac{X^{i+1}_{\varphi(t_n)}(\varphi(u_{i+1}),\cdot)+X^{i+1}_{\varphi(t_n)}(\varphi(v_{i+1}),\cdot)}{2}, \ (a.e.) \end{aligned}$$

Integrating the above inequality on Δ^i with respect to $\varphi(t_i)$ for each $i = 1, 2, ..., n-1$, then almost everywhere

$$\begin{aligned} &\frac{1}{\Delta^i_-}\int_{\varphi(u_i)}^{\Delta^i_+ -\varphi(u_i)} X^{i+1}_{\varphi(t_n)}\left(\frac{\Delta^{i+1}_+}{2},\cdot\right)d\varphi(t_i) \\ &\le \frac{1}{\Delta^i_-\ \Delta^{i+1}_-}\int_{\varphi(u_i)}^{\Delta^i_+ -\varphi(u_i)}\int_{\varphi(u_{i+1})}^{\Delta^{i+1}_+ -\varphi(u_{i+1})} X^{i+1}_{\varphi(t_n)}(\varphi(t_{i+1}),\cdot)\ d\varphi(t_{i+1})d\varphi(t_i) \\ &\le \frac{1}{2\Delta^i_-}\int_{\varphi(u_i)}^{\Delta^i_+ -\varphi(u_i)}\left(X^{i+1}_{\varphi(t_n)}(\varphi(u_{i+1}),\cdot)+X^{i+1}_{\varphi(t_n)}(\varphi(v_{i+1}),\cdot)\right)d\varphi(t_i) \end{aligned} \tag{2.3}$$

Applying HHII to the left hand of (2.3)

$$\begin{aligned} &X\left(\left(\bigwedge_{k=1}^{i-1}\varphi(t_k),\ \frac{\Delta_+^i}{2},\frac{\Delta_+^{i+1}}{2},\bigwedge_{k=i+2}^{n}\varphi(t_k)\right),\cdot\right) \\ &\leq \frac{1}{\Delta_-^i}\int_{\varphi(u_i)}^{\Delta_+^i-\varphi(u_i)} X_{\varphi(t_n)}^{i+1}\left(\frac{\Delta_+^{i+1}}{2},\cdot\right)d\varphi(t_i) \end{aligned} \tag{2.4}$$

and also applying HHII to the right hand of (2.3)

$$\begin{aligned} &\frac{1}{2}\left[\frac{1}{\Delta_-^i}\int_{\varphi(u_i)}^{\Delta_+^i-\varphi(u_i)} X_{\varphi(t_n)}^{i+1}(\varphi(u_{i+1}),\cdot)d\varphi(t_i)+\frac{1}{\Delta_-^i}\int_{\varphi(u_i)}^{\Delta_+^i-\varphi(u_i)} X_{\varphi(t_n)}^{i+1}(\varphi(v_{i+1}),\cdot)d\varphi(t_i)\right] \\ &\leq \frac{1}{4}\left[\begin{array}{l} X\left(\left(\bigwedge_{k=1}^{i-1}\varphi(t_k),\varphi(u_i),\quad \varphi(u_{i+1}),\bigwedge_{k=i+2}^{n}\varphi(t_k)\right),\cdot\right) \\ +X\left(\left(\bigwedge_{k=1}^{i-1}\varphi(t_k),\varphi(v_i),\quad \varphi(u_{i+1}),\bigwedge_{k=i+2}^{n}\varphi(t_k)\right),\cdot\right) \\ +X\left(\left(\bigwedge_{k=1}^{i-1}\varphi(t_k),\varphi(u_i),\quad \varphi(v_{i+1}),\bigwedge_{k=i+2}^{n}\varphi(t_k)\right),\cdot\right) \\ +X\left(\left(\bigwedge_{k=1}^{i-1}\varphi(t_k),\varphi(v_i),\quad \varphi(v_{i+1}),\bigwedge_{k=i+2}^{n}\varphi(t_k)\right),\cdot\right) \end{array}\right] \end{aligned} \tag{2.5}$$

for each $i=1,2,\ldots,n-1$. After using the inequalities (2.4) and (2.5) in (2.3), then taking summation from 1 to $n-1$, we have (2.2).

Theorem 2.3. Let $X:\mathfrak{D}^n\times\Omega\to\mathbb{R}_+$ be $\mathrm{MGP}_{\eta\varphi}\mathrm{SP}$ with respect to η and φ on $\mathfrak{D}^n$. If the assumptions of Lemma 2.1 satisfy, then almost everywhere

$$\begin{aligned} &\sum_{i=1}^{n}\frac{1}{2\Delta_-^i}\int_{\varphi(u_i)}^{\Delta_+^i-\varphi(u_i)}\left(X_{\varphi(u_n)}^i\ (\varphi(t_i),\cdot)+X_{\varphi(v_n)}^i\ (\varphi(t_i),\cdot)\right)d\varphi(t_i) \\ &\leq \frac{n}{2}[X(\varphi(\boldsymbol{u}),\cdot)+X(\varphi(\boldsymbol{v}),\cdot)]+\frac{1}{2}\sum_{i=1}^{n}\left[X_{\varphi(u_n)}^i\ (\varphi(v_i),\cdot)+X_{\varphi(v_n)}^i\ (\varphi(u_i),\cdot)\right] \end{aligned} \tag{2.6}$$

Proof. Taking into account Remark 2.1 for $X_{\varphi(u_n)}^i$ and $X_{\varphi(v_n)}^i$, $i=1,2,\ldots,n$, respectively, then almost everywhere

$$\begin{aligned} &\frac{1}{\Delta_-^i}\int_{\varphi(u_i)}^{\Delta_+^i-\varphi(u_i)} X_{\varphi(u_n)}^i\ (\varphi(t_i),\cdot)d\varphi(t_i)\ \leq \frac{X_{\varphi(u_n)}^i\ (\varphi(u_i),\cdot)+X_{\varphi(u_n)}^i\ \left(\Delta_+^i\ -\varphi(u_i),\cdot\right)}{2} \\ &\leq \frac{X_{\varphi(u_n)}^i\ (\varphi(u_i),\cdot)+X_{\varphi(u_n)}^i\ (\varphi(v_i),\cdot)}{2}\leq \frac{X(\varphi(\boldsymbol{u}),\cdot)+X_{\varphi(u_n)}^i\ (\varphi(v_i),\cdot)}{2}; \\ &\frac{1}{\Delta_-^i}\int_{\varphi(u_i)}^{\Delta_+^i-\varphi(u_i)} X_{\varphi(v_n)}^i\ (\varphi(t_i),\cdot)d\varphi(t_i)\ \leq \frac{X_{\varphi(v_n)}^i\ (\varphi(u_i),\cdot)+X_{\varphi(v_n)}^i\ \left(\Delta_+^i\ -\varphi(u_i),\cdot\right)}{2} \\ &\leq \frac{X_{\varphi(v_n)}^i\ (\varphi(u_i),\cdot)+X_{\varphi(v_n)}^i\ (\varphi(v_i),\cdot)}{2}\leq \frac{X_{\varphi(v_n)}^i\ (\varphi(u_i),\cdot)+X(\varphi(\boldsymbol{v}),\cdot)}{2} \end{aligned}$$

Adding the above inequalities, we have

$$\frac{1}{\Delta_-^i}\int_{\varphi(u_i)}^{\Delta_+^i-\varphi(u_i)}\left[X^i_{\varphi(u_n)}(\varphi(t_i),\cdot)+X^i_{\varphi(v_n)}(\varphi(t_i),\cdot)\right]d\varphi(t_i)$$
$$\leq\frac{X(\varphi(\boldsymbol{u}),\cdot)+X(\varphi(\boldsymbol{v}),\cdot)+X^i_{\varphi(u_n)}(\varphi(v_i),\cdot)+X^i_{\varphi(v_n)}(\varphi(u_i),\cdot)}{2}$$

Adding above n inequalities, we get (2.6).

Theorem 2.4. Let $X:\mathfrak{D}^n\times\Omega\to\mathbb{R}_+$ be MG$P_{\eta\varphi}$SP with respect to η and φ on $\mathfrak{D}^n$. If the assumptions of Lemma 2.1 satisfy, then almost everywhere

$$\begin{aligned}&X\left(\left(\frac{\Delta_+^1}{2},\dots,\frac{\Delta_+^n}{2}\right),\cdot\right)\\ &\leq\frac{1}{\prod_{i=1}^{n}\Delta_-^i}\int_{\varphi(u_1)}^{\Delta_+^1-\varphi(u_1)}\cdots\int_{\varphi(u_n)}^{\Delta_+^n-\varphi(u_n)}X\big((\varphi(t_1),\ \dots,\varphi(t_n)),\cdot\big)d\varphi(t_n)\dots d\varphi(t_1)\\ &\leq\frac{1}{2^n}\sum_{\boldsymbol{\delta}\in m_i(n)}X\big((\boldsymbol{\delta}\varphi(\boldsymbol{u})+(1-\boldsymbol{\delta})\varphi(\boldsymbol{v})),\cdot\big)\end{aligned}\tag{2.7}$$

where $m_i(n):=\{\boldsymbol{\delta}:=(\delta_1,\dots,\delta_n)\in\mathbb{N}_0^n:\ \delta_i\leq 1;\ |\boldsymbol{\delta}|=n+1-i;\ i=1,\dots,n+1\};\ |\boldsymbol{\delta}|:=\delta_1+\dots+\delta_n;\ \boldsymbol{\delta}\varphi(\boldsymbol{u}):=(\delta_1\varphi(u_1),\dots,\delta_n\varphi(u_n))$.

Proof. Taking into account Remark 2.1 for $X^n_{\varphi(t_n)}$, then we get the following inequality

$$\begin{aligned}X^n_{\varphi(t_n)}\left(\frac{\Delta_+^n}{2},\cdot\right)&\leq\frac{1}{\Delta_-^n}\int_{\varphi(u_n)}^{\Delta_+^n-\varphi(u_n)}X^n_{\varphi(t_n)}(\varphi(t_n),\cdot)d\varphi(t_n)\\ &\leq\frac{X^n_{\varphi(t_n)}(\varphi(u_n),\cdot)+X^n_{\varphi(t_n)}(\varphi(v_n),\cdot)}{2},\ (a.e.)\end{aligned}\tag{2.8}$$

Integrating (2.8) on Δ^{n-1}, then we obtain almost everywhere

$$\begin{aligned}&\frac{1}{\Delta_-^{n-1}}\int_{\varphi(u_{n-1})}^{\Delta_+^{n-1}-\varphi(u_{n-1})}X^n_{\varphi(t_n)}\left(\frac{\Delta_+^n}{2},\cdot\right)d\varphi(t_{n-1})\\ &\leq\frac{1}{\Delta_-^{n-1}\Delta_-^n}\int_{\varphi(u_{n-1})}^{\Delta_+^{n-1}-\varphi(u_{n-1})}\int_{\varphi(u_n)}^{\Delta_+^n-\varphi(u_n)}X^n_{\varphi(t_n)}(\varphi(t_n),\cdot)d\varphi(t_n)d\varphi(t_{n-1})\\ &\leq\frac{1}{\Delta_-^{n-1}}\int_{\varphi(u_{n-1})}^{\Delta_+^{n-1}-\varphi(u_{n-1})}\left(\frac{X^n_{\varphi(t_n)}(\varphi(u_n),\cdot)+X^n_{\varphi(t_n)}(\varphi(v_n),\cdot)}{2}\right)d\varphi(t_{n-1})\end{aligned}\tag{2.9}$$

Using HHII for the left hand of (2.9) and the right hand of (2.9), respectively

$$\begin{aligned}&X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\frac{\Delta_+^{n-1}}{2},\frac{\Delta_+^n}{2}\right),\cdot\right)\\ &\leq\frac{1}{\Delta_-^{n-1}}\int_{\varphi(u_{n-1})}^{\Delta_+^{n-1}-\varphi(u_{n-1})}X^n_{\varphi(t_n)}\left(\frac{\Delta_+^n}{2},\cdot\right)d\varphi(t_{n-1})\end{aligned}\tag{2.10}$$

$$\frac{1}{\Delta_{-}^{n-1}}\int_{\varphi(u_{n-1})}^{\Delta_{+}^{n-1}-\varphi(u_{n-1})}\left(\frac{X^{n}_{\varphi(t_n)}\ (\varphi(u_n),\cdot)+X^{n}_{\varphi(t_n)}\ (\varphi(v_n),\cdot)}{2}\right)d\varphi(t_{n-1})$$
$$=\frac{1}{2}\left[\begin{array}{l}\frac{1}{\Delta_{-}^{n-1}}\int_{\varphi(u_{n-1})}^{\Delta_{+}^{n-1}-\varphi(u_{n-1})}X^{n}_{\varphi(t_n)}\ (\varphi(u_n),\cdot)d\varphi(t_{n-1})\\ +\frac{1}{\Delta_{-}^{n-1}}\int_{\varphi(u_{n-1})}^{\Delta_{+}^{n-1}-\varphi(u_{n-1})}X^{n}_{\varphi(t_n)}\ (\varphi(v_n),\cdot)d\varphi(t_{n-1})\end{array}\right] \tag{2.11}$$
$$\leq\frac{1}{2^2}\left[\begin{array}{l}X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \varphi(u_{n-1}),\varphi(u_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \varphi(v_{n-1}),\varphi(u_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \varphi(u_{n-1}),\varphi(v_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \varphi(v_{n-1}),\varphi(v_n)\right),\cdot\right)\end{array}\right]$$

From (2.9)–(2.11)

$$X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \frac{\Delta_{+}^{n-1}}{2},\frac{\Delta_{+}^{n}}{2}\right),\cdot\right)$$
$$\leq\frac{1}{\Delta_{-}^{n-1}\Delta_{-}^{n}}\int_{\varphi(u_{n-1})}^{\Delta_{+}^{n-1}-\varphi(u_{n-1})}\int_{\varphi(u_n)}^{\Delta_{+}^{n}-\varphi(u_n)}X^{n}_{\varphi(t_n)}\ (\varphi(t_n),\cdot)d\varphi(t_n)d\varphi(t_{n-1}) \tag{2.12}$$
$$\leq\frac{1}{2^2}\left[\begin{array}{l}X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \varphi(u_{n-1}),\varphi(u_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \varphi(v_{n-1}),\varphi(u_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \varphi(u_{n-1}),\varphi(v_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \varphi(v_{n-1}),\varphi(v_n)\right),\cdot\right)\end{array}\right]$$

Integrating (2.12) on Δ^{n-2}

$$\frac{1}{\Delta_{-}^{n-2}}\int_{\varphi(u_{n-2})}^{\Delta_{+}^{n-2}-\varphi(u_{n-2})}X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\frac{\Delta_{+}^{n-1}}{2},\frac{\Delta_{+}^{n}}{2}\right),\cdot\right)d\varphi(t_{n-2})\leq\frac{1}{\prod_{i=n-2}^{n}\Delta_{-}^{i}}$$
$$\times\int_{\varphi(u_{n-2})}^{\Delta_{+}^{n-2}-\varphi(u_{n-2})}\int_{\varphi(u_{n-1})}^{\Delta_{+}^{n-1}-\varphi(u_{n-1})}\int_{\varphi(u_n)}^{\Delta_{+}^{n}\ -\varphi(u_n)}X^{n}_{\varphi(t_n)}\ (\varphi(t_n),\cdot)d\varphi(t_n)d\varphi(t_{n-1})d\varphi(t_{n-2})$$
$$\leq\frac{1}{2^2\Delta_{-}^{n-2}}\int_{\varphi(u_{n-2})}^{\Delta_{+}^{n-2}-\varphi(u_{n-2})}\left[\begin{array}{l}X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \varphi(u_{n-1}),\varphi(u_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \varphi(v_{n-1}),\varphi(u_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \varphi(u_{n-1}),\varphi(v_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \ \varphi(v_{n-1}),\varphi(v_n)\right),\cdot\right)\end{array}\right]d\varphi(t_{n-2}) \tag{2.13}$$

Similarly, using the above method by (2.13)

$$\frac{1}{\Delta_-^{n-2}}\int_{\varphi(u_{n-2})}^{\Delta_+^{n-2}-\varphi(u_{n-2})} X\left(\left(\bigwedge_{k=1}^{n-2}\varphi(t_k),\ \frac{\Delta_+^{n-1}}{2},\frac{\Delta_+^{n}}{2}\right),\cdot\right)d\varphi(t_{n-2}) \leq \frac{1}{\prod_{i=n-2}^{n}\Delta_-^i}$$

$$\times\int_{\varphi(u_{n-2})}^{\Delta_+^{n-2}-\varphi(u_{n-2})}\int_{\varphi(u_{n-1})}^{\Delta_+^{n-1}-\varphi(u_{n-1})}\int_{\varphi(u_n)}^{\Delta_+^{n}-\varphi(u_n)} X^n_{\varphi(t_n)}\ (\varphi(t_n),\cdot)d\varphi(t_n)d\varphi(t_{n-1})d\varphi(t_{n-2})$$

$$\leq\frac{1}{2^3}\left[\begin{array}{l} X\left(\left(\bigwedge_{k=1}^{n-3}\varphi(t_k),\ \varphi(u_{n-2}),\varphi(u_{n-1}),\varphi(u_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-3}\varphi(t_k),\varphi(v_{n-2}),\varphi(u_{n-1}),\varphi(u_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-3}\varphi(t_k),\varphi(u_{n-2}),\varphi(v_{n-1}),\varphi(u_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-3}\varphi(t_k),\varphi(v_{n-2}),\varphi(v_{n-1}),\varphi(u_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-3}\varphi(t_k),\varphi(u_{n-2}),\varphi(u_{n-1}),\varphi(v_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-3}\varphi(t_k),\varphi(v_{n-2}),\varphi(u_{n-1}),\varphi(v_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-3}\varphi(t_k),\varphi(u_{n-2}),\varphi(v_{n-1}),\varphi(v_n)\right),\cdot\right)\\ +X\left(\left(\bigwedge_{k=1}^{n-3}\varphi(t_k),\varphi(v_{n-2}),\varphi(v_{n-1}),\varphi(v_n)\right),\cdot\right)\end{array}\right]$$

Implementing the above procedure and we obtain (2.7) by $X^n_{\varphi(t_n)}\ (\varphi(t_n),\cdot) := X\big(\varphi(t_1),\ \dots,\varphi(t_n),\cdot\big)$.

Remark 2.2. Taking into account Theorem 2.4, under the suitable conditions, then one can obtain HHII for the stochastic processes in Definition 2.4.

Example 2.2. Let $X:\mathfrak{D}^2\times\Omega\to\mathbb{R}_+$ be a two-dimensional $GP_{\eta\varphi}SP$ with respect to η and φ on $\mathfrak{D}^2$. Then almost everywhere

$$X\left(\left(\frac{2\varphi(u_1)+\eta(\varphi(v_1),\varphi(u_1))}{2},\frac{2\varphi(u_2)+\eta(\varphi(v_2),\varphi(u_2))}{2}\right),\cdot\right)$$
$$\leq\frac{1}{\eta(\varphi(v_1),\varphi(u_1))\eta(\varphi(v_2),\varphi(u_2))}$$
$$\times\int_{\varphi(u_1)}^{\varphi(u_1)+\eta(\varphi(v_1),\varphi(u_1))}\int_{\varphi(u_2)}^{\varphi(u_2)+\eta(\varphi(v_2),\varphi(u_2))}X((\varphi(t_1),\varphi(t_2)),\cdot)d\varphi(t_2)d\varphi(t_1)$$
$$\leq\frac{1}{2^3}\left[\begin{array}{l}X((\varphi(u_1),\varphi(u_2))),\cdot)+X((\varphi(v_1),\varphi(u_2))),\cdot)\\ +X((\varphi(u_1),\varphi(v_2))),\cdot)+X((\varphi(v_1),\varphi(v_2))),\cdot)\end{array}\right]$$

Really, according to Theorem 2.4 for $n=2$, we get $m_i(2):=\big\{\boldsymbol{\delta}:=(\delta_1,\delta_2)\in\mathbb{N}_0^2:\ \delta_i\leq 1,\ |\boldsymbol{\delta}|=3-i\big\},\ i=1,\ 2,\ 3.$ Namely $m_1(2)=\big\{(1,1)\big\};\ m_2(2)=\big\{(0,1),(1,0)\big\};\ m_3(2)=\big\{(0,0)\big\}$. Thus

$$\sum_{\boldsymbol{\delta}\in m_1(2)} X\big((\boldsymbol{\delta}\varphi(\boldsymbol{u})+(1-\boldsymbol{\delta})\varphi(\boldsymbol{v})),\cdot\big) = X((\varphi(u_1),\varphi(u_2)),\cdot)$$
$$= X\big(((1,1)(\varphi(u_1),\varphi(u_2)) + [(1,1)-(1,1)](\varphi(v_1),\varphi(v_2))),\cdot\big)$$

for $\varphi(\boldsymbol{u}) = (\varphi(u_1),\varphi(u_2))$, $\varphi(\boldsymbol{v}) = (\varphi(v_1),\varphi(v_2))$ and $m_1(2) = \{(1,1)\}$. Similarly, for $m_2(2) = \{(0,1),(1,0)\}$ and $m_3(2) = \{(0,0)\}$, respectively

$$\sum_{\boldsymbol{\delta}\in m_2(2)} X\big((\boldsymbol{\delta}\varphi(\boldsymbol{u})+(1-\boldsymbol{\delta})\varphi(\boldsymbol{v})),\cdot\big)$$
$$= X((\varphi(v_1),\varphi(u_2)),\cdot) + X((\varphi(u_1),\varphi(v_2)),\cdot);$$
$$\sum_{\boldsymbol{\delta}\in m_3(2)} X\big((\boldsymbol{\delta}\varphi(\boldsymbol{u})+(1-\boldsymbol{\delta})\varphi(\boldsymbol{v})),\cdot\big) = X((\varphi(v_1),\varphi(v_2)),\cdot)$$

Therefore

$$\sum_{\boldsymbol{\delta}\in m_i(2)} X\big((\boldsymbol{\delta}\varphi(\boldsymbol{u})+(1-\boldsymbol{\delta})\varphi(\boldsymbol{v})),\cdot\big) = X((\varphi(u_1),\varphi(u_2)),\cdot) + X((\varphi(v_1),\varphi(u_2)),\cdot)$$
$$+X((\varphi(u_1),\varphi(v_2)),\cdot) + X((\varphi(v_1),\varphi(v_2)),\cdot)$$

Finally, we obtained the desired inequality for two-dimensional general preinvex stochastic processes using the above equalities in Theorem 2.1.

3. Conclusion

The fundamental contribution of this study has been the introduction of stochastic general preinvexity for processes on real number line and on the coordinates, respectively. Also, we showed some characteristics of these processes. We verified some Hermite–Hadamard inequalities for all of these processes. We hope that original results can be obtained for different processes using the methods in this study.

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Competing interests

The author declares that he has no competing interests.

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