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Hermite-Hadamard Type Inequality for Log-preinvex Functions via Sugeno Integrals

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Abstract. In this paper, it is showed firstly that classical Hermite-Hadamard type inequality is not satisfied for log-preinvex functions based on Sugeno integrals. Moreover, an upper bound on the Sugeno fuzzy integral of log-preinvex functions is established on a discrete set. Finally, an upper bound on the Sugeno fuzzy integral of log-preinvex functions on general form is obtained as an alternative to classical Hermite-Hadamard type inequality.

1. Introduction

Aggregation is a process of combining several numerical values into a single one which exists in many disciplines, such as image processing, pattern recognition and decision making. To obtain a consensus quantifiable judgment, some synthesizing functions have been proposed. For example, arithmetic mean, geometric mean and median can be regarded as a basic class, because they are often used and very classic. However, these operators are not able to model an interaction between criteria. For having a representation of interaction phenomena between criteria, fuzzy measures have been proposed by Sugeno [1] in 1974.

In recent years, some authors generalized several classical integral inequalities for fuzzy integral. Li, Song and Yue [2] served Hermite-Hadamard type inequality for Sugeno integrals in 2014. Caballero and Sadarangani [3] showed Hermite-Hadamard type inequality of fuzzy integrals for convex function in 2009. A stronger property of convexity is log-convexity.

The arithmetic mean-geometric mean inequality easily yields that every log-convex function is also convex. The behavior of certain interference-coupled multiuser systems can be modeled by means of logarithmically convex interference functions by Boche [4] in 2008.

In 2015 Abbaszadeh [5] estimated the upper bound of Sugeno fuzzy integral for log-convex functions using the classical Hadamard integral inequality. Turhan et all [6] obtained Hermite-Hadamard type inequality for strongly convex functions via Sugeno Integrals in 2017.

A significant generalization of convex functions is that of invex functions introduced by Hanson [7] in 1981. Hanson's initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Weir and Mond [8] and Noor [9, 10, 11] have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems in 1988, 2005, 2006. Sarikaya et all [12] obtained Hermite-Hadamard Type Integral Inequalities for preinvex and log-preinvex functions in 2013. Moreover, Turhan et all [13] extended to Sugeno Integrals Sarikayas' results [12] on Hermite-Hadamard type inequality for preinvex functions in 2015.

In the light of these developments, our main goal is to prove Hermite-Hadamard type inequality for log-preinvex functions based on Sugeno integrals that is an important extension of log-convexity.

Let's see some proporties of fuzzy integral.

2. Preliminary Discussions

In this section, we remember some basic definition and properties of fuzzy integral and log-preinvex function. For details we refer the readers to Refs [1, 2, 10].

Definition 2.1. *Suppose that Σ is a σ-algebra of subsets of X and that $\mu : \Sigma \to [0,\infty)$ is a non-negative, extended real-valued set function. We say that μ is a fuzzy measure if and only if*

1. $\mu(\emptyset) = 0$;
2. $E, F \in \Sigma$ *and* $E \subset F$ *imply* $\mu(E) \le \mu(F)$;
3. $\{E_n\} \subset \Sigma, E_1 \subset E_2 \subset ...,$ *imply* $\lim\limits_{n\to\infty} \mu(E_n) = \mu\left(\bigcup\limits_{n=1}^{\infty} E_n\right)$;
4. $\{E_n\} \subset \Sigma, E_1 \supset E_2 \supset ..., \mu(E_1) < \infty,$ *imply* $\lim\limits_{n\to\infty} \mu(E_n) = \mu\left(\bigcap\limits_{n=1}^{\infty} E_n\right)$.

Then the triple (X, Σ, μ) is called a fuzzy measure space.

Let (X, Σ, μ) be a fuzzy measure space. By $\Im_+(X)$, we denote the set

$$\Im_+(X) = \{ f : X \to [0,\infty) | \, f \text{ is measurable with respect to } \Sigma \}.$$

For $f \in \Im_+(X)$ and $\alpha > 0$, F_α and $F_{\tilde{\alpha}}$ we will denote the following sets:

$$\begin{aligned} F_\alpha &= \{f \ge \alpha\} = \{x \in X : \ f(x) \ge \alpha\}; \\ F_{\tilde{\alpha}} &= \{f > \alpha\} = \{x \in X : \ f(x) > \alpha\}. \end{aligned}$$

Note that if $\alpha \le \beta$, then $F_\beta \subset F_\alpha$ and $F_{\tilde{\beta}} \subset F_{\tilde{\alpha}}$.

Definition 2.2. *Let* $(X,\ \Sigma,\ \mu)$ *be a fuzzy measure space,* $f \in \Im_{+}(X)$ *and* $A \in \Sigma$*, then the Sugeno fuzzy integral of* f *on* A *with respect to* μ *which is defined by*

$$(s)\int_A f\ d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)).$$

When $A = X$*, the fuzzy integral may also be denoted by* $(s)\int_A f\ d\mu$*. Where* $\vee$ *and* $\wedge$ *denote the operations infimum and supremum on* $(0,\infty)$*, respectively.*

The following properties of the Sugeno integral are well known and can be found in.

Theorem 2.1. *Let* $(X,\ \Sigma,\ \mu)$ *be a fuzzy measure space,* $f,g \in \Im_{+}(X)$ *and* $A \in \Sigma$ *then*

1. $(s)\int_A f\ d\mu \leq \mu(A)$;
2. $(s)\int_A f\ d\mu = k \wedge \mu(A)$*,* k *non-negative constant;*
3. *If* $f \leq g$ *on* A *then* $(s)\int_A f\ d\mu \leq (s)\int_A g\ d\mu$;
4. *If* $A \subset B$*, then* $(s)\int_A f\ d\mu \leq (s)\int_B f\ d\mu$;
5. $\mu(A \cap F_\alpha) \geq \alpha \Rightarrow (s)\int_A f\ d\mu \geq \alpha$;
6. $\mu(A \cap F_\alpha) \leq \alpha \Rightarrow (s)\int_A f\ d\mu \leq \alpha$.

Remark 2.1. *Consider the distribution function* F *associated to* f *on* A*, that is,* $F(\alpha) = \mu(A \cap F_\alpha)$*. Then, due to (5) and (6) of Theorem 2.1, we have that*

$$F(\alpha) = \alpha \Rightarrow (s)\int_A f\ d\mu = \alpha.$$

Thus, from a numerical point of view, the fuzzy integral can be calculated solving the equation $F(\alpha) = \alpha$*.*

Definition 2.3. *A non-empty closed subset* I *of* $\mathbb{R}^n$ *is said to be invex set with respect to the given vector function* $\eta : I \times I \to \mathbb{R}^n$*, if* $x + \lambda\eta(y,x) \in I$ *for all* $x, y \in I$ *and* $\lambda \in [0,1]$*. Clearly, any convex set is an invex set with respect to* $\eta(y,x) = y - x$*.*

Definition 2.4. *Let* $I \subseteq R^n$ *be an invex set with respect to* $\eta : I \times I \to \mathbb{R}^n$*. Then the function (not necessarily differentiable)* $f : I \to \mathbb{R}$ *is said to be preinvex with respect to* η *if*

$$f(x + \lambda\eta(y,x)) \leq (1-\lambda)f(x) + \lambda f(y)$$

for all $x, y \in I$ *and* $\lambda \in [0,1]$*. Any convex function is preinvex with respect to* $\eta(y,x) = y - x$*. but the converse is not necessarily true.*

Condition C. Let $\eta : I \times I \to \mathbb{R}$. It is told that the function η satisfies Condition C if,

$$\begin{array}{l}(C1)\ \eta(x, x+\lambda\eta(y,x)) = -\lambda\eta(y,x)\\ (C2)\ \eta(y, x+\lambda\eta(y,x)) = (1-\lambda)\eta(y,x)\end{array}$$

for all $u, v \in I$ and $\lambda \in [0,1]$. Additionally, note that from condition C, we have

$$\eta(x+\lambda_2\eta(y,x), x+\lambda_1\eta(y,x)) = (\lambda_2-\lambda_1)\eta(y,x)$$

for all $x, y \in I$ and $\lambda_1, \lambda_2 \in [0,1]$.

Definition 2.5. *Let $I \subseteq R^n$ be an invex set with respect to $\eta : I \times I \to \mathbb{R}^n$. Then the function (not necessarily differentiable) $f : I \to \mathbb{R}$ is said to be log-preinvex with respect to η if*

$$f\left(x+\lambda\eta\left(y,x\right)\right) \ \leq \ f\left(x\right)^{(1-\lambda)} f\left(y\right)^{\lambda}$$

for all $x, y \in I$ and $\lambda \in [0,1]$. Any log-convex function is log-preinvex with respect to $\eta(y,x) = y - x$ but the converse is not necessarily true.

The following theorem shows that the classical Hermite-Hadamard inequality for log-preinvex functions:

Theorem 2.2. *Let $f : [a, a+\eta(b,a)] \subset I \to (0,\infty)$ be a log-preinvex function on the interval of the real number I°(the interior of I) and $a, b \in I^{\circ}$ with $a+\eta(b,a) \leq b$. Then the following inequality holds:*

$$\frac{1}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)} f\left(x\right)dx \leq L\left(f\left(a\right), f\left(b\right)\right).$$

It will be convenient to invoke the logarithmic mean $L\left(f\left(a\right), f\left(b\right)\right)$ of two positive numbers $f\left(a\right), f\left(b\right)$, which is given by

$$L\left(f\left(a\right), f\left(b\right)\right) = \begin{cases} \frac{f(a)-f(b)}{\ln(f(a))-\ln(f(b))}, & f\left(a\right) \neq f\left(b\right) \\ f\left(a\right), & f\left(a\right) = f\left(b\right) \end{cases}.$$

3. Main Results

In this section, the aim of study is to obtain the Hermite-Hadamard inequality for log-preinvex functions via Sugeno fuzzy integrals. Firstly, we assume that (X, Σ, μ) is a general fuzzy measure space. To simplify the calculation of the fuzzy integral, for a given $f \in \Im_{+}(X)$ and $A \in \Sigma$, we write

$$\Gamma = \{\alpha \mid \alpha \geq 0, \mu(A \cap F_\alpha) > \mu(A \cap F_\beta) \text{ for any } \beta > \alpha\}.$$

It is easy to see that

$$(s)\int_A f\, d\mu = \bigvee_{\alpha \in \Gamma} (\alpha \wedge \mu(A \cap F_\alpha)).$$

The following example shows that the Hermite-Hadamard inequality for log-preinvex function is not valid in the fuzzy context.

Example 3.1. Consider $X = [0, \eta(1,0)]$ and let μ be the Lebesgue measure on X. If we take the function $f(x) = e^{-x}$, then $f(x)$ is a log-convex function. To calculate the Sugeno integral related to this function, let's consider the distribution function F associated to f to $[0, \eta(1,0)]$, by Remark 2.1, this is

$$\begin{aligned}(s)\int_0^{\eta(1,0)} e^{-x} d\mu &= \bigvee_{\alpha \geq 0} \left(\alpha \wedge \mu\left([0, \eta(1,0)] \cap \left\{e^{-x} \geq \alpha\right\}\right)\right) \\ &= \bigvee_{\alpha \geq 0} (\alpha \wedge \mu([0, \eta(1,0)] \cap \{x \leq -\ln(\alpha)\})) \\ &= \bigvee_{\alpha \geq 0} (\alpha \wedge (-\ln \alpha)).\end{aligned}$$

In this expression, $(-\ln \alpha)$ may be negative, but it is a decreasing continuous function of α when $\alpha \geq 0$.Hence, the supremum will be attained at the point which is one of the solutions of the equation $-\ln \alpha = \alpha$, that is, at $\alpha \approx 0,5672$. So we have

$$(s)\int_0^{\eta(1,0)} f d\mu \approx 0,5672.$$

On the other hand, $L(f(0), f(\eta(1,0))) \approx 0,3231$. This proves that the Hermite-Hadamard inequality is not satisfied in the fuzzy context.

Now, we will establish an upper bound on the Sugeno fuzzy integral of log-preinvex functions. A specific example will be given to illustrate the result.

Theorem 3.1. *Let $f : [0, \eta(1,0)] \to (0, \infty)$ be a log-preinvex function such that $f(0) \neq f(\eta(1,0))$ and μ the Lebesque measure on $\mathbb{R}$. Then*

$$(s)\int_0^{\eta(1,0)} f d\mu \leq \bigvee_{\alpha \in \Gamma} \left(\alpha \wedge \mu\left([0, \eta(1,0)] \cap \left\{[f(0)]^{1-\frac{x}{\eta(1,0)}} . [f(\eta(1,0))]^{\frac{x}{\eta(1,0)}} \geq \alpha\right\}\right)\right),$$

where

$$\begin{aligned}\Gamma &= [f(0), f(\eta(1,0)))\ \textit{for}\ f(\eta(1,0)) > f(0);\\ \Gamma &= [f(\eta(1,0)), f(0))\ \textit{for}\ f(\eta(1,0)) < f(0).\end{aligned}$$

Proof. Using the log-preinvexity of f, we have

$$\begin{aligned}f(x) &= f\left(x.0 + \frac{x}{\eta(1,0)}\eta(1,0)\right)\\ &\leq [f(0)]^{1-\frac{x}{\eta(1,0)}} . [f(\eta(1,0))]^{\frac{x}{\eta(1,0)}} = g(x)\end{aligned}$$

for $x \in [0, \eta(1,0)]$. Hence, by (3) of Theorem 2.1, we get

$$\begin{aligned}(s)\int_0^{\eta(1,0)} f d\mu &\leq (s)\int_0^{\eta(1,0)} [f(0)]^{1-\frac{x}{\eta(1,0)}} . [f(\eta(1,0))]^{\frac{x}{\eta(1,0)}} d\mu\\ &= (s)\int_0^{\eta(1,0)} g d\mu.\end{aligned}$$

In order to calculate the integral in the right-hand part of the last inequality, we consider the distribution function G given by

$$G(\alpha) = \mu([0, \eta(1,0)] \cap \{g \geq \alpha\}).$$

If $f(\eta(1,0)) > f(0)$, then

$$\begin{aligned}G(\alpha) &= \mu\left([0,\eta(1,0)] \cap \left\{[f(0)]^{1-\frac{x}{\eta(1,0)}} . [f(\eta(1,0))]^{\frac{x}{\eta(1,0)}} \geq \alpha\right\}\right)\\ &= \mu\left([0,\eta(1,0)] \cap \left\{x \geq \log_{\frac{f(\eta(1,0))}{f(0)}} \frac{\alpha}{f(0)}\right\}\right)\\ &= \mu\left(\left[\log_{\frac{f(\eta(1,0))}{f(0)}} \frac{\alpha}{f(0)}, \eta(1,0)\right]\right).\end{aligned}$$

Thus $\Gamma = [f(0), f(\eta(1,0)))$, and we only need to consider $\alpha \in [f(0), f(\eta(1,0)))$.

If $f(\eta(1,0)) < f(0)$, then

$$G(\alpha) = \mu\left([0,\eta(1,0)] \cap \left\{[f(0)]^{1-\frac{x}{\eta(1,0)}} . [f(\eta(1,0))]^{\frac{x}{\eta(1,0)}} \geq \alpha\right\}\right)$$

$$= \mu\left([0,\eta(1,0)] \cap \left\{x \geq \log_{\frac{f(0)}{f(\eta(1,0))}} \frac{f(0)}{\alpha}\right\}\right)$$

$$= \mu\left(\left[0, \log_{\frac{f(0)}{f(\eta(1,0))}} \frac{f(0)}{\alpha}\right]\right).$$

Thus $\Gamma = [f(\eta(1,0)), f(0))$, and we only need to consider $\alpha \in [f(\eta(1,0)), f(0))$.

Taking into account (3) Theorem 2.1, we get

$$(s)\int_0^{\eta(1,0)} g d\mu = \bigvee_{\alpha\in\Gamma} (\alpha \wedge G(\alpha)) \geq (s)\int_0^{\eta(1,0)} f d\mu$$

and the proof is completed.

Remark 3.1. *In the case $f(0) = f(\eta(1,0))$ in Theorem 3.2, the function $g(x)$ is*

$$g(x) = [f(0)]^{1-\frac{x}{\eta(1,0)}} . [f(\eta(1,0))]^{\frac{x}{\eta(1,0)}} = f(0)$$

and (3) Theorem 2.1, we get

$$\begin{aligned}
(s)\int_0^{\eta(1,0)} f d\mu &\leq (s)\int_0^{\eta(1,0)} g d\mu \\
&= (s)\int_0^{\eta(1,0)} f(0)\, d\mu \\
&= f(0) \wedge \mu([0,\eta(1,0)]).
\end{aligned}$$

Finally, we can obtain as as an alternative to the classical Hermite-Hadamard type inequality an upper bound on the Sugeno fuzzy integral of log-preinvex functions on general form.

Theorem 3.2. *Suppose that $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a log-preinvex function with $f(a) \neq f(a+\eta(b,a))$. Then*

$$(s)\int_a^{a+\eta(b,a)} f\, d\mu \leq \bigvee_{\alpha\in\Gamma} (\alpha \wedge \mu([a, a+\eta(b,a)] \cap \{g(t) \geq \alpha\})),$$

where

$$\begin{aligned} g(t) &= f(a)^{1-t} f(a+\eta(b,a))^{t} \ \ for\ t=\frac{x-a}{\eta(b,a)}; \\ \Gamma &= [f(a), f(a+\eta(b,a)))\, for f(a+\eta(b,a)) > f(a); \\ \Gamma &= [f(a+\eta(b,a)), f(a))\, for f(a) > f(a+\eta(b,a)). \end{aligned}$$

Proof. As f log-preinvexity, for $x \in [a, a+\eta(b,a)]$, we have

$$\begin{aligned} f(x) &= f\left(a+\left(\frac{x-a}{\eta(b,a)}\right).\eta(b,a)\right) \\ &\leq f(a)^{1-t} f(a+\eta(b,a))^{t} = g(t) \end{aligned}$$

for all $t=\frac{x-a}{\eta(b,a)}$.

By (3) in Theorem 2.1, we get

$$\begin{aligned} (s)\int_a^{a+\eta(b,a)} f\, d\mu &\leq (s)\int_a^{a+\eta(b,a)} f(a)^{1-t} f(a+\eta(b,a))^{t}\ d\mu \\ &= (s)\int_a^{a+\eta(b,a)} g\, d\mu\,. \end{aligned}$$

As similar argument as in the proof Theorem 3.1 yields

$$(s)\int_a^{a+\eta(b,a)} g\, d\mu \leq \bigvee_{\alpha\in\Gamma} \left(\alpha \wedge \mu\left([a, a+\eta(b,a)] \cap \{g(t) \geq \alpha\}\right)\right),$$

where

$$\begin{aligned} g(t) &= f(a)^{1-t} f(a+\eta(b,a))^{t} \ \ for\ t=\frac{x-a}{\eta(b,a)}; \\ \Gamma &= [f(a), f(a+\eta(b,a)))\, for f(a+\eta(b,a)) > f(a); \\ \Gamma &= [f(a+\eta(b,a)), f(a))\, for f(a) > f(a+\eta(b,a)). \end{aligned}$$

This completes the proof.

Remark 3.2. *In the case $f(a) = f(a+\eta(b,a))$ in Theorem 3.2, we have $g(x) = f(a)$, and by (3) in Theorem 2.1, we get*

$$\begin{aligned} (s)\int_a^{a+\eta(b,a)} f\, d\mu &\leq (s)\int_a^{a+\eta(b,a)} g\, d\mu \\ &= (s)\int_a^{a+\eta(b,a)} f(a)\ d\mu \\ &= f(a) \wedge \mu([a, a+\eta(b,a)]). \end{aligned}$$

4. Conclusion

In this paper, we have established an upper bound on the Sugeno fuzzy integral of log-preinvex functions which is a useful tool to estimate unsolvable integrals of this kind. In many applications, assumptions about the log-convexity of a probability distribution allow just enough special structure to yield a workable theory. The log-concavity or log-convexity of probability densities and their integrals has interesting qualitative implications in many areas of economics, in political science, in biology, and in industrial engineering. As we know, fuzzy measures have been introduced by Sugeno in the early seventies in order to extend probability measures by relaxing the additivity property. Thus the study of the Sugeno fuzzy integral for log-preinvex functions is an important and interesting topic for further research.

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