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On the existence of nonoscillatory solutions of three-dimensional time scale systems

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To My Ph.D. Advisor, Prof. Dr. Elvan Akın.

Abstract. We consider a three-dimensional nonlinear system of first order dynamic equations on time scales and show the existence and asymptotic behavior of nonoscillatory solutions by using the most well-known fixed point theorems. Examples are also provided, which validates our theoretical claims.

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1. Introduction

In this paper, we study the nonlinear system

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = b(t)g(z(t)) \\ z^{\Delta}(t) = c(t)h(x(t)), \end{cases} \tag{1.1}$$

where $a, b, c \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$, and f and g are nondecreasing functions such that $uf(u) > 0$, $ug(u) > 0$ and $uh(u) > 0$ for $u \neq 0$. The theory of time scales, denoted by $\mathbb{T}$, started by Stefan Hilger in his Ph.D thesis not only unifies continuous and discrete analyses but also extends the results in one comprehensive theory and eliminates obscurity from both. Afterwards, the theory and advances of time scales are published in a series of two books by Bohner and Peterson in 2001 and 2003, see [3,4]. Throughout this paper, we assume that $\mathbb{T}$ is unbounded above and whenever we write $t \geq t_0$, we mean $t \in [t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$.

Nonoscillation plays a very important role in the theory of 3D systems of first-order dynamic equations on time scales to have enough information about the behavior of nonoscillatory solutions in a long term. Some applica-

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tions of such systems in discrete and continuous cases arise in control theory, stability theory and models for flows of thin viscous films over solid surfaces. For example, for $\mathbb{T} = \mathbb{R}$, Bernis and Peletier [2] considered an equation, which can be rewritten as a system:

$$\begin{cases} u_1 = u_2 \\ u_2 = u_3 \\ u_3 = h(u), \end{cases}$$

in order to discuss existence, uniqueness and qualitative properties of solutions for flows of thin viscous films over solid surfaces. The other continuous and discrete versions of system (1.1) were studied by Chanturia [6] and Schmeidel [15,16], respectively.

A solution (x, y, z) of system (1.1) is said to be proper if

$$\sup\{|x(s)|, |y(s)|, |z(s)| : s \in [t, \infty)_{\mathbb{T}}\} > 0$$

for $t \geq t_0$. A proper solution (x, y, z) of (1.1) is said to be nonoscillatory if the component functions x, y and z are nonoscillatory, i.e., either eventually positive or eventually negative. Otherwise it is said to be oscillatory.

For the sake of simplicity, let us set

$$A(t_0, t) = \int_{t_0}^{t} a(s)\Delta s, \quad B(t_0, t) = \int_{t_0}^{t} b(s)\Delta s \quad \text{and} \quad C(t_0, t) = \int_{t_0}^{t} c(s)\Delta s,$$

where $s, t, t_0 \in \mathbb{T}$ and throughout the paper, we assume that $A(t_0, \infty) = B(t_0, \infty) = \infty$.

Suppose that S is the set of all nonoscillatory solutions (x, y, z) of system (1.1). Then it was shown in [1] that any nonoscillatory solution (x, y, z) of system (1.1) belongs to one of the following classes:

$$S^+ := \{(x, y, z) \in S : \operatorname{sgn} x(t) = \operatorname{sgn} y(t) = \operatorname{sgn} z(t),\ t \geq t_0\}$$
$$S^- := \{(x, y, z) \in S : \operatorname{sgn} x(t) = \operatorname{sgn} y(t) \neq \operatorname{sgn} z(t),\ t \geq t_0\}.$$

In the literature, solutions in S^+ and S^- are known as *Type* (a) and *Type* (c) solutions, respectively. We refer the reader to the article by Akin, Došla and Lawrence [1] for the proofs of the following lemmas:

Lemma 1.1. *Suppose that (x, y, z) is a nonoscillatory solution of system (1.1).*

(i) *Any nonoscillatory solution of (1.1) in S^+ satisfies*

$$\lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \infty.$$

(ii) *Any nonoscillatory solution of (1.1) in S^- satisfies*

$$\lim_{t\to\infty} |z(t)| = 0.$$

We give the following lemma that provides the criteria for relatively compactness, see [8].

Lemma 1.2. *Suppose that $X \subseteq BC[t_0, \infty)_{\mathbb{T}}$ is bounded and uniformly Cauchy. Further, suppose that X is equi-continuous on $[t_0, t_1]_{\mathbb{T}}$ for any $t_1 \in [t_0, \infty)_{\mathbb{T}}$. Then X is relatively compact.*

We give the Schauder fixed point theorem, proved by Juliusz Schauder in 1930 and the Knaster fixed-point theorem, see [10,17], respectively.

Theorem 1.3. (Schauder's Fixed Point Theorem) *Let M be a nonempty, closed, bounded, convex subset of a Banach space X, and suppose that $T : M \to M$ is a compact operator. Then, T has a fixed point.*

Theorem 1.4. (Knaster Fixed Point Theorem) *If $(M, \leq)$ is a complete lattice and $T : M \to M$ is order-preserving (also called monotone or isotone), then T has a fixed point. In fact, the set of fixed points of T is a complete lattice.*

The remainder of the paper is organized as follows: in Sect. 2, we demonstrate the existence of nonoscillatory solutions of system (1.1) in S^+ and S^- by using certain improper integrals. In Sect. 3, we provide several examples to highlight our main results. Finally, we give open problems and a conclusion in the last section.

2. Existence in S^+ and S^-

In this section, we do not only show the asymptotic properties of nonoscillatory solutions of system (1.1), but also the existence of such solutions in S^+ and S^- by using certain improper integrals via fixed point theorems. Set

$$\begin{aligned}
Y_1 &= \int_{t_0}^{\infty} c(t)h\left(\int_{t_0}^{t} a(s)f\left(k_1\int_{t_0}^{s} b(\tau)\Delta\tau\right)\Delta s\right)\Delta t,\\
Y_2 &= \int_{t_0}^{\infty} a(t)f\left(k_2+\int_{t}^{\infty} b(s)g\left(k_3\int_{s}^{\infty} c(\tau)\Delta\tau\right)\Delta s\right)\Delta t,\\
Y_3 &= \int_{t_0}^{\infty} b(t)g\left(\int_{t}^{\infty} c(s)h\left(k_4\int_{t_0}^{s} a(\tau)\Delta\tau\right)\Delta s\right)\Delta t,\\
Y_4 &= \int_{t_0}^{\infty} a(t)f\left(\int_{t}^{\infty} b(s)g\left(k_5\int_{s}^{\infty} c(\tau)\Delta\tau\right)\Delta s\right)\Delta t,
\end{aligned}$$

for some $k_1, k_2, k_3, k_4, k_5 \neq 0$.

2.1. Existence in S^+

Suppose that (x, y, z) is a nonoscillatory solution of system (1.1) in S^+ such that $x > 0$ eventually. (The case $x < 0$ can be shown similarly.) Then by the first, second and third equations of system (1.1), we have that x, y and z are positive increasing functions. Then we conclude that $x \to c_1$ or $x \to \infty$, $y \to c_2$ or $y \to \infty$, and $z \to c_3$ or $z \to \infty$, where $0 < c_1, c_2, c_3 < \infty$. However, the case $x(t) \to c_1$ and $y(t) \to c_2$ cannot occur because of Lemma 1.1 (i). Therefore, in view of this information, we have the following lemma:

Lemma 2.1. *Let (x, y, z) be a nonoscillatory solution of system (1.1). Then such a solution belongs to one of the following subclasses:*

$$S^+_{\infty,\infty,B} := \{(x, y, z) \in S^+ : \lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \infty,\ \lim_{t\to\infty} |z(t)| = c_3\}$$
$$S^+_{\infty,\infty,\infty} := \{(x, y, z) \in S^+ : \lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |z(t)| = \infty\}.$$

Theorem 2.2. *$S^{+}_{\infty,\infty,B} \neq \emptyset$ if $Y_1 < \infty$.*

Proof. Suppose that $Y_1 < \infty$. Then choose $t_1 \geq t_0$, $k_1 > 0$ such that

$$\int_{t_1}^{\infty} c(t)h\left(\int_{t_1}^{t} a(s)f\left(k_1\int_{t_1}^{s} b(\tau)\Delta\tau\right)\Delta s\right)\Delta t < d, \quad t \geq t_1, \tag{2.1}$$

where $k_1 = g(2d)$. Let X be the partially ordered Banach space of all real-valued continuous functions with the norm $\|z\| = \sup_{t\geq t_1}|z(t)|$ and the usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega := \{z \in X : d \leq z(t) \leq 2d,\ t \geq t_1\}$$

and an operator $Fz : X \to X$ by

$$(Fz)(t) = d + \int_{t_1}^{t} c(s)h\left(\int_{t_1}^{s} a(u)f\left(\int_{t_1}^{u} b(\tau)g(z(\tau))\Delta\tau\right)\Delta u\right)\Delta s \tag{2.2}$$

for $t \geq t_1$. First, it is easy to see that F is increasing, so let us show that $Fx : \Omega \to \Omega$. Indeed,

$$d \leq (Fz)(t) \leq d + \int_{t_1}^{t} c(s)h\left(\int_{t_1}^{s} a(u)f\left(\int_{t_1}^{u} b(\tau)g(2d)\Delta\tau\right)\Delta u\right)\Delta s \leq 2d$$

by (2.1). Also, it is easy to show that $\inf B \in \Omega$ and $\sup B \in \Omega$ for any subset B of Ω, i.e., $(\Omega, \leq)$ is a complete lattice. Therefore, by Theorem 1.4, see [10], we have that there exists $\bar{z} \in \Omega$ such that $\bar{z} = F\bar{z}$, i.e.,

$$\bar{z}(t) = d + \int_{t_1}^{t} c(s)h\left(\int_{t_1}^{s} a(u)f\left(\int_{t_1}^{u} b(\tau)g(\bar{z}(\tau))\Delta\tau\right)\Delta u\right)\Delta s. \tag{2.3}$$

Then by taking the derivative of (2.3) we have

$$\bar{z}^{\Delta}(t) = c(t)h\left(\int_{t_1}^{t} a(u)f\left(\int_{t_1}^{u} b(\tau)g(\bar{z}(\tau))\Delta\tau\right)\Delta u\right), \quad t \geq t_1.$$

Setting

$$\bar{x}(t) = \int_{t_1}^{t} a(u)f\left(\int_{t_1}^{u} b(\tau)g(\bar{z}(\tau))\Delta\tau\right)\Delta u \tag{2.4}$$

and taking the derivative yields

$$\bar{x}^{\Delta}(t) = a(t)f\left(\int_{t_1}^{t} b(\tau)g(\bar{z}(\tau))\Delta\tau\right), \quad t \geq t_1.$$

Finally, letting

$$\bar{y}(t) = \int_{t_1}^{t} b(\tau)g(\bar{z}(\tau))\Delta\tau \tag{2.5}$$

and taking the derivative gives

$$\bar{y}^{\Delta}(t) = b(t)g(\bar{z}(t)), \quad t \geq t_1,$$

which implies that $(\bar{x}, \bar{y}, \bar{z})$ is a solution of system (1.1). By taking the limit of (2.3)–(2.5) as $t \to \infty$, we have that $\bar{x}(t) \to \infty$, $\bar{y}(t) \to \infty$ and $\bar{z}(t) \to \alpha$, where $0 < \alpha < \infty$, i.e., $S^{+}_{\infty,\infty,B} \neq \emptyset$. $\square$

Since it is not easy to find a sufficient condition for the existence of nonoscillatory solutions in $S^{+}_{\infty,\infty,\infty}$, we have the following theorem by assuming the existence of such solutions in S^{+}. We leave the proof to readers.

Theorem 2.3. *Let (x, y, z) be a nonoscillatory solution of system (1.1). Then every solution in S^{+} belongs to $S^{+}_{\infty,\infty,\infty}$ if $C(t_0, \infty) = \infty$.*

2.2. Existence in S^{-}

This section represents the limit behavior of nonoscillatory solutions of system (1.1) along with the existence of such solutions in S^{-}. So suppose that (x, y, z) is a nonoscillatory solution of system (1.1) such that $x > 0$ eventually. By the same discussion as in Sect. 2.1 and by Lemma 1.1 (ii), one can have the following lemma:

Lemma 2.4. *Assume that (x, y, z) is a nonoscillatory solution of system (1.1) in S^{-}. Then (x, y, z) belongs to one of the following subclasses:*

$$\begin{aligned}
S^{-}_{B,B,0} &:= \{(x, y, z) \in S^{-} : \lim_{t\to\infty} |x(t)| = c_1, \lim_{t\to\infty} |y(t)| = c_2, \lim_{t\to\infty} |z(t)| = 0\} \\
S^{-}_{B,0,0} &:= \{(x, y, z) \in S^{-} : \lim_{t\to\infty} |x(t)| = c_1 \lim_{t\to\infty} |y(t)| = 0, \lim_{t\to\infty} |z(t)| = 0\} \\
S^{-}_{\infty,B,0} &:= \{(x, y, z) \in S^{-} : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |y(t)| = c_2, \lim_{t\to\infty} |z(t)| = 0\} \\
S^{-}_{\infty,0,0} &:= \{(x, y, z) \in S^{-} : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |y(t)| = 0, \lim_{t\to\infty} |z(t)| = 0\},
\end{aligned}$$

where $0 < c_1, c_2 < \infty$.

Theorem 2.5. *$S^{-}_{B,B,0} \neq \emptyset$ if $Y_2 < \infty$, provided g is an odd function.*

Proof. Suppose that $Y_2 < \infty$. Then, we can choose $k_2, k_3 > 0$ and $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} a(t) f\left(k_2 + \int_{t}^{\infty} b(s) g\left(k_3 \int_{s}^{\infty} c(\tau)\Delta\tau\right)\Delta s\right)\Delta t < d_1, \tag{2.6}$$

where $k_3 = h(2d)$. Let X be the space of all real-valued continuous and bounded functions with the norm $\|x\| = \sup_{t\geq t_1} |x(t)|$. It is clear that X is a Banach space, see [7]. Define a subset Ω of X such that

$$\Omega := \{x \in X : d_1 \leq x(t) \leq 2d_1, \, t \geq t_1\}.$$

Let us define an operator $Fx : X \to \Omega$ such that

$$(Fx)(t) = d_1 + \int_{t_1}^{t} a(s) f\left(d_2 + \int_{s}^{\infty} b(u) g\left(\int_{u}^{\infty} c(\tau) h(x(\tau))\Delta\tau\right)\Delta u\right)\Delta s.$$

It is clear that Ω is closed, bounded and convex. First we show that $Fx : \Omega \to \Omega$. Indeed,

$$\begin{aligned}
d_1 \leq (Fx)(t) &\leq d_1 + \int_{t_1}^{t} a(s) f\left(d_2 + \int_{s}^{\infty} b(u) g\left(h(2d_1) \int_{u}^{\infty} c(\tau)\Delta\tau\right)\Delta u\right)\Delta s \\
&\leq 2d_1.
\end{aligned}$$

Next, we need to show F is continuous on Ω. Let x_n be a sequence in Ω such that $x_n \to x \in \Omega = \overline{\Omega}$. Since

$$\begin{aligned}&\|(Fx_n)(t) - (Fx)(t)\|\\ &\le \int_{t_1}^{t} a(s)\left| f\left(d_2 + \int_s^\infty b(u)g\left(\int_u^\infty c(\tau)h(x_n(\tau))\Delta\tau\right)\Delta u\right)\right.\\ &\quad \left. - f\left(d_2 + \int_s^\infty b(u)g\left(\int_u^\infty c(\tau)h(x(\tau))\Delta\tau\right)\Delta u\right)\right|\Delta s.\end{aligned}$$

Then the Lebesgue dominated convergence theorem and the continuity of f, g and h imply that F is continuous on Ω. Finally, we need to show that F is relatively compact, i.e., equibounded and equicontinuous. Note that

$$(Fx)^{\Delta}(t) = a(t)f\left(d_2 + \int_t^\infty b(u)g\left(\int_u^\infty c(\tau)h(x(\tau))\Delta\tau\right)\Delta u\right) < \infty,$$

since f is a real-valued function. Then by Lemma 1.2 and the mean value theorem, it follows that F is relatively compact. Therefore, there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$ by Theorem 1.3, see [17]. Also it is clear that $\bar{x}(t)$ converges to a finite number as $t \to \infty$. By the same discussion as in Theorem 2.2 and setting

$$\bar{y}(t) = d_2 + \int_t^\infty b(u)g\left(\int_u^\infty c(\tau)h(\bar{x}(\tau))\Delta\tau\right)\Delta u > 0, \quad t \ge t_1,$$

and

$$\bar{z}(t) = -\int_t^\infty c(\tau)h(\bar{x}(\tau))\Delta\tau < 0, \quad t \ge t_1,$$

we get $\bar{y}(t) \to d_2$ and $\bar{z}(t) \to 0$. So we conclude that $(\bar{x}, \bar{y}, \bar{z})$ is a nonoscillatory solution of system (1.1) in $S^-_{B,B,0}$. □

Theorem 2.6. *$S^-_{\infty,B,0} \neq \emptyset$ if $Y_3 < \infty$, provided g is an odd function.*

Proof. Suppose that $Y_3 < \infty$. Then choose $k_3 > 0$ and a large enough $t_1 \ge t_0$ such that

$$\int_{t_1}^\infty b(t)g\left(\int_t^\infty c(s)h\left(k_3\int_{t_1}^s a(\tau)\Delta\tau\right)\Delta s\right)\Delta t < \frac{1}{2}, \tag{2.7}$$

where $k_3 = f(1)$. Let X be the partially ordered Banach space of all real-valued continuous functions with the norm $\|y\| = \sup_{t \ge t_1} |y(t)|$ and the usual pointwise ordering $\le$. Define a subset Ω of X such that

$$\Omega := \left\{y \in X : \frac{1}{2} \le y(t) \le 1,\ t \ge t_1\right\},$$

and an operator $Fy : X \to X$ by

$$(Fy)(t) = \frac{1}{2} + \int_t^\infty b(s)g\left(\int_s^\infty c(u)h\left(\int_{t_1}^u a(\tau)f(y(\tau))\Delta\tau\right)\Delta u\right)\Delta s. \tag{2.8}$$

First we need to show $(\Omega, \leq)$ is a complete lattice. Indeed, $\inf B \in \Omega$ and $\sup B \in \Omega$ for any subset B of Ω, i.e., $(\Omega, \leq)$ is a complete lattice. It is also clear that

$$\frac{1}{2} \leq (Fy)(t) \leq 1,$$

by (2.7), i.e., $Fy : \Omega \to \Omega$. Finally, it can be shown that F is an increasing mapping. Therefore, by the Knaster fixed point theorem, there exists $\bar{y} \in \Omega$ such that $\bar{y} = F\bar{y}$. So we have that $\bar{y}(t) > 0$ for $t \geq t_1$ and converges to $\frac{1}{2}$ as $t \to \infty$. Since g is odd, setting

$$\bar{z}(t) = -\int_t^\infty c(u)h\left(\int_{t_1}^u a(\tau)f(\bar{y}(\tau))\Delta\tau\right)\Delta u, \tag{2.9}$$

and

$$\bar{x}(t) = \int_{t_1}^t a(\tau)f(\bar{y}(\tau))\Delta\tau \tag{2.10}$$

and taking the limits of (2.9) and (2.10) as $t \to \infty$ gives $\bar{z}(t) \to 0$ and $\bar{x}(t) \to \infty$, i.e., $S^-_{\infty,B,0} \neq \emptyset$. □

Since the following theorem can be similarly shown as in Theorem 2.5, the proof is omitted:

Theorem 2.7. *$S^-_{B,0,0} \neq \emptyset$ if $Y_4 < \infty$, provided g is an odd function.*

Theorem 2.8. *$S^-_{\infty,0,0} \neq \emptyset$ if $Y_3 < \infty$ and $Y_4 = \infty$, provided g is an odd function.*

Proof. Suppose that $Y_3 < \infty$ and $Y_4 = \infty$. Then choose $t_1 \geq t_0$ and $k_4, k_5 > 0$ such that

$$\int_{t_1}^\infty b(t)g\left(\int_t^\infty c(s)h\left(k_4\int_{t_1}^s a(\tau)\Delta\tau\right)\Delta s\right)\Delta s < \frac{1}{2}, \quad t \geq t_1, \tag{2.11}$$

and

$$\int_{t_1}^\infty a(t)f\left(\int_t^\infty b(s)g\left(k_5\int_s^\infty c(\tau)\Delta\tau\right)\Delta s\right)\Delta t > \frac{1}{2}, \quad t \geq t_1, \tag{2.12}$$

where $k_4 = \frac{1}{2}$ and $k_5 = h(\frac{1}{2})$. Let X be the partially ordered Banach space of all continuous functions with the supremum norm $\|x\| = \sup\limits_{t\geq t_1} \frac{x(t)}{A(t_1,t)}$ and usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega := \left\{x \in X : \frac{1}{2} \leq x(t) \leq \frac{1}{2}\int_{t_1}^t a(s)\Delta s,\, t \geq t_1\right\}$$

and an operator $Fx : X \to X$ by

$$(Fx)(t) = \int_{t_1}^t a(s)f\left(\int_s^\infty b(\tau)g\left(\int_\tau^\infty c(u)h(x(u))\Delta u\right)\Delta s\right)\Delta t.$$

By the same argument as in Theorem 2.6, it can be shown that $(\Omega, \leq)$ is a complete lattice and $F : \Omega \to \Omega$ is an increasing mapping. So by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. So $\bar{x}(t) \to \infty$ as $t \to \infty$. By setting

$$\bar{y}(t) = \int_t^\infty b(\tau) g\left(\int_\tau^\infty c(u)h(\bar{x}(u))\Delta u\right)\Delta\tau, \quad t \geq t_1$$

and

$$\bar{z}(t) = -\int_t^\infty c(u)h(\bar{x}(u))\Delta u, \quad t \geq t_1,$$

we have $\bar{y}(t) > 0$ and $\bar{z}(t) < 0$ for $t \geq t_1$ such that $\bar{y}(t) \to 0$ and $\bar{z}(t) \to 0$ as $t \to \infty$. This completes the proof. □

3. Examples

In this section, we provide a few examples for our results given in Sect. 2 for an important time scale. In order to do that, we give the following proposition which provides us enough information about how we define the derivative and integral on the specific time scale that we use in our examples. We refer the reader to the book written by Bohner and Peterson [3] for more information.

Definition 3.1. Let $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Then we introduce the forward jump operator $\sigma(t) = tq$, the backward jump operator $\rho(t) = \frac{t}{q}$ and the graininess function $\mu(t) = (q-1)t$ for $t \in \mathbb{T}$. We also define the delta derivative of a function $p : \mathbb{T} \to \mathbb{R}$ by

$$p^\Delta(t) = \frac{p(\sigma(t)) - p(t)}{\mu(t)}$$

for $t \in \mathbb{T}$ and the integral by

$$\int_a^b p(t)\Delta t = \sum_{t \in [a, \rho(b)]_{q^{\mathbb{N}_0}}} p(t)\mu(t) \tag{3.1}$$

for $a, b \in \mathbb{T}$.

Example. Let $\mathbb{T} = q^{\mathbb{N}_0}$, $a(t) = \frac{1}{(1+t)^{\frac{1}{3}}}$, $b(t) = \left(\frac{t}{2t-1}\right)^{\frac{1}{5}}$, $c(t) = \frac{1}{qt^3}$, $f(u) = u^{\frac{1}{3}}$, $g(u) = u^{\frac{1}{5}}$, $h(u) = u$, $s = q^m$, $t = q^n$, $k_1 = 1$ and $t_0 = 1$. We show that $S^+_{\infty,\infty,B} \neq \emptyset$ by Theorem 2.2. So we need to show $A(t_0, \infty) = B(t_0, \infty) = \infty$ and $Y_1 < \infty$. Indeed,

$$\int_1^T a(t)\Delta t = \sum_{t \in [1, \rho(T)]_{q^{\mathbb{N}_0}}} \frac{1}{(1+t)^{\frac{1}{3}}} \cdot (q-1)t.$$

So as $T \to \infty$, we have

$$A(1, \infty) = (q-1)\sum_{n=0}^\infty \frac{q^n}{(1+q^n)^{\frac{1}{3}}} = \infty$$

by the ratio test. Similarly, one can show $B(1,\infty)=\infty$. Finally, let us show $Y_1<\infty$:

$$\begin{aligned}
&\int_1^T c(t)h\left(\int_1^t a(s)f\left(\int_1^s b(\tau)\Delta\tau\right)\Delta s\right)\Delta t\\
&\quad=\int_1^T c(t)h\left(\int_1^t a(s)\left(\sum_{\tau\in[1,\rho(s)]_{q^{\mathbb{N}_0}}}\left(\frac{t}{2t-1}\right)^{\frac{1}{5}}\cdot(q-1)t\right)^{\frac{1}{3}}\Delta s\right)\Delta t\\
&\quad\le\int_1^T c(t)h\left(\int_1^t a(s)\cdot s^{\frac{1}{3}}\right)\Delta t\\
&\quad=(q-1)\int_1^T c(t)\left(\sum_{s\in[1,\rho(t)]_{q^{\mathbb{N}_0}}}\frac{1}{(1+s)^{\frac{1}{3}}}\cdot s^{\frac{1}{3}}\cdot s\right)\Delta t\\
&\quad\le(q-1)\sum_{t\in[1,\rho(T)]_{q^{\mathbb{N}_0}}}\frac{1}{t}.
\end{aligned}$$

Hence, taking the limit of the latter inequality as $T\to\infty$ gives us

$$\sum_{n=0}^{\infty}\frac{1}{q^n}<\infty,$$

by the geometric series. Therefore, $Y_1<\infty$ by the comparison test. It can also be shown that $(t,1+t,2-\frac{1}{t})$ is a nonoscillatory solution of

$$\begin{cases}
x^{\Delta}(t)=\frac{1}{(1+t)^{\frac{1}{3}}}y^{\frac{1}{3}}(t)\\
y^{\Delta}(t)=\left(\frac{t}{2t-1}\right)^{\frac{1}{5}}z^{\frac{1}{5}}(t)\\
z^{\Delta}(t)=\frac{1}{qt^3}x(t),
\end{cases}$$

such that $x(t)\to\infty, y(t)\to\infty$ and $z(t)\to 2$ as $t\to\infty$, i.e., $S^{+}_{\infty,\infty,B}\neq\emptyset$.

Example. Let $\mathbb{T}=2^{\mathbb{N}_0}$, $a(t)=\left(\frac{t}{3t+1}\right)^{\frac{1}{3}}$, $b(t)=\frac{1}{2}$, $c(t)=\frac{3}{4(1+t)t^3}$, $f(u)=u^{\frac{1}{3}}$, and $g(u)=h(u)=u$. One can easily show $A(t_0,\infty)=B(t_0,\infty)=\infty$. So let us show $Y_3<\infty$ for $k_4=1$ and $t_0=1$. First note that

$$\int_1^s a(\tau)\Delta\tau=\sum_{\tau\in[1,\rho(s)]_{2^{\mathbb{N}_0}}}\left(\frac{\tau}{3\tau+1}\right)^{\frac{1}{3}}\cdot\tau\le\left(\frac{1}{3}\right)^{\frac{1}{3}}\sum_{\tau\in[1,\rho(s)]_{2^{\mathbb{N}_0}}}\tau\le s. \tag{3.2}$$

Then by (3.1) and (3.2), we have

$$\int_t^T c(s)h\left(\int_1^s a(\tau)\Delta\tau\right)\Delta s\le\sum_{s\in[t,\rho(T)]_{2^{\mathbb{N}_0}}}\frac{3}{4(1+s)s}\le\sum_{s\in[t,\rho(T)]_{2^{\mathbb{N}_0}}}\frac{1}{s^2}.$$

Then as $T\to\infty$, the latter inequality gives us

$$\int_t^{\infty} c(s)h\left(\int_1^s a(\tau)\Delta\tau\right)\Delta s\le\frac{4}{3t^2}. \tag{3.3}$$

Finally, we have by (3.3) that

$$\int_1^T b(t)g\left(\int_t^\infty c(s)h\left(\int_{t_0}^s a(\tau)\Delta\tau\right)\Delta s\right)\Delta t \leq \sum_{t\in[1,\rho(T))_{2^{\mathbb{N}_0}}} \frac{2}{3t}.$$

Therefore, we have $Y_3 < \infty$ as $T \to \infty$ by the geometric series. One can show that $(1+t, \frac{3t+1}{t}, \frac{-1}{t^2})$ is a solution of

$$\begin{cases} x^\Delta(t) = \left(\frac{t}{3t+1}\right)^{\frac{1}{3}} y^{\frac{1}{3}}(t) \\ y^\Delta(t) = \frac{1}{2}z(t) \\ z^\Delta(t) = \frac{3}{4(1+t)t^3}x(t), \end{cases}$$

in S^- such that $x(t) \to \infty, y(t) \to 3$ and $z(t) \to 0$ as $t \to \infty$, i.e., $S^-_{\infty,B,0} \neq \emptyset$ by Theorem 2.6.

Example. Let $\mathbb{T} = 2^{\mathbb{N}_0}, a(t) = \frac{t-1}{t^2}, b(t) = \frac{t^2}{(2t-1)(t-1)}, c(t) = \frac{3}{4(4t-1)t^2}, f(u) = g(u) = h(u) = u$, $k_5 = 1$ and $t_0 = 3$. By an idea analogous to the previous examples, it can be shown that $A(t_0,\infty) = B(t_0,\infty) = \infty$ and $Y_4 < \infty$. Then by Theorem 2.7, there exists a nonoscillatory solution of system (1.1) in $S^-_{B,0,0}$. Indeed, $(\frac{4t-1}{t}, \frac{1}{2t-2}, \frac{-1}{t^2})$ is a solution of

$$\begin{cases} x^\Delta(t) = \frac{t-1}{t^2}y(t) \\ y^\Delta(t) = \frac{t^2}{(2t-1)(t-1)}z(t) \\ z^\Delta(t) = \frac{3}{4(4t-1)t^2}x(t) \end{cases}$$

in S^- such that $x(t) \to 4$, $y(t) \to 0$ and $z(t) \to 0$ as $t \to \infty$.

4. Conclusion and open problems

This article has presented a novel approach to the existence of nonoscillatory solutions of system (1.1) in S^+ and S^- for the case $A(t_0,\infty) = B(t_0,\infty) = \infty$. Showing the (non)existence of nonoscillatory solutions for higher order time scale systems is avoided because of the difficulty of setting the operators for the fixed point theorems. Hence, we do not only find the integral criteria for the existence but we guarantee that there exists such solutions by utilizing suitable fixed point theorems. For future work, we plan to consider the systems

$$\begin{cases} x^\Delta(t) = a(t)f(y(t)) \\ y^\Delta(t) = b(t)g(z(t)) \\ z^\Delta(t) = -c(t)h(x(t)) \end{cases} \tag{4.1}$$

and

$$\begin{cases} x^\Delta(t) = a(t)|y(t)|^\alpha \operatorname{sgn} y(t) \\ y^\Delta(t) = b(t)|z(t)|^\beta \operatorname{sgn} z(t) \\ z^\Delta(t) = \lambda c(t)|x^\sigma(t)|^\gamma \operatorname{sgn} x^\sigma(t), \end{cases} \tag{4.2}$$

where $\alpha, \beta, \gamma > 0$ are real numbers and $\lambda = \pm 1$. We intend to show the existence and nonexistence of nonoscillatory solutions in S^+ and S^- and provide new results for oscillation criteria. System (4.2) for $\lambda = -1$ is known as a three-dimensional Emden–Fowler time scale system, see [5,9]. Two-dimensional Emden–Fowler time scale systems, which have several applications in astrophysics, gas dynamics, fluid mechanics, relativistic mechanics, nuclear physics and chemically reacting systems, were considered by Öztürk and Akın [11–14].

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