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Limiting Behaviors of Nonoscillatory Solutions for Two-Dimensional Nonlinear Time Scale Systems

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Abstract. We consider a two-dimensional time scale system of first order dynamic equations and establish some necessary and sufficient conditions for the existence of nonoscillatory solutions for the system using Knaster fixed point theorem, the Schauder fixed point theorem and the Schauder–Tychonoff fixed point theorem. We also provide examples to underline the main results of this article.

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1. Introduction

In this paper, we deal with the classification schemes of nonoscillatory solutions of the system

$$\begin{cases} x^{\Delta}(t) = p(t)f(y(t)) \\ y^{\Delta}(t) = -r(t)g(x^{\sigma}(t)), \end{cases} \tag{1.1}$$

where $p, r \in \mathrm{C_{rd}}\left([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+\right)$ and $f, g \in \mathrm{C}(\mathbb{R},\mathbb{R})$ are nondecreasing such that $uf(u) > 0$, $ug(u) > 0$ for $u \neq 0$. The time scale theory was first introduced by Stephen Hilger in his PhD thesis in 1988 that does not only unify discrete and continuous analysis but also extend the results for all time scales and eliminate the obscurity from both. A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers and the theory of time scales was published in two books by Bohner and Peterson in [2] and [3]. We assume that $\mathbb{T}$ is unbounded above throughout this paper. Whenever we write $t \geq t_1$, we mean that $t \in [t_1,\infty)_{\mathbb{T}} := [t_1,\infty) \cap \mathbb{T}$. We call (x, y) a *proper solution* if it is defined on $[t_0,\infty)_{\mathbb{T}}$ and $sup\{|x(s)|, |y(s)| : s \in [t,\infty)_{\mathbb{T}}\} > 0$ for $t \geq t_0$. If $\mathbb{T} = \mathbb{R}$, then system (1.1) is reduced to a two-dimensional system of first order differential equations, see [7]. The continuous case but more general system is also considered in [4].

A solution (x, y) of (1.1) is said to be nonoscillatory if the component functions x and y are both nonoscillatory, i.e., either eventually positive or eventually negative. Otherwise, it is said to be oscillatory. It is well known that if (x, y) is a nonoscillatory solution of system (1.1), then the component functions x and y are themselves nonoscillatory, see [1].

Let M be the set of all nonoscillatory solutions of system (1.1). One can easily show that any nonoscillatory solution (x, y) of system (1.1) belongs to one of the following classes:

$$M^+ := \{(x, y) \in M : \ xy > 0 \quad \text{eventually}\}$$
$$M^- := \{(x, y) \in M : \ xy < 0 \quad \text{eventually}\}.$$

For convenience, let us set

$$P(t, s) = \int_t^s p(u)\Delta u \quad \text{and} \quad R(t, s) = \int_t^s r(u)\Delta u$$

In Sect. 2, we show the existence of nonoscillatory solutions of system (1.1) in M^+ using certain improper integrals based on the convergence/divergence of $P(t_0, \infty)$ and $R(t_0, \infty)$, $t_0 \in \mathbb{T}$. In order to do that, we use well known fixed point theorems such as the Knaster fixed point theorem, see [6] the Schauder fixed point theorem and Tychonoff fixed point theorem. In Sect. 3, we classify nonoscillatory solutions for system (1.1) in M^- and we also reduce system (1.1) into a special case, which is known as Emden–Fowler systems in the literature see [8] and focus on the existence in M^- since the results for M^+ have already been known, see [9].

The following lemma shows the oscillation and nonoscillation criteria for system (1.1), which can be proven as in [10] and [11].

Lemma 1.1. (a) *If $P(t_0, \infty) < \infty$ and $R(t_0, \infty) < \infty$, then system (1.1) is nonoscillatory.*

(b) *If $P(t_0, \infty) = \infty$ and $R(t_0, \infty) = \infty$, then system (1.1) is oscillatory.*
(c) *If $P(t_0, \infty) = \infty$ and $R(t_0, \infty) < \infty$, then any nonoscillatory solution (x, y) of system (1.1) belongs to M^+, i.e, $M^- = \emptyset$.*
(d) *If $P(t_0, \infty) < \infty$ and $R(t_0, \infty) = \infty$, then any nonoscillatory solution (x, y) of system (1.1) belongs to M^-, i.e., $M^+ = \emptyset$.*
(e) *If $P(t_0, \infty) < \infty$, then the component function x has a finite limit.*
(f) *If $P(t_0, \infty) = \infty$ or $R(t_0, \infty) < \infty$, then the component function y has a finite limit.*
(g) *Let $P(t_0, \infty) = \infty$ and (x, y) be a nonoscillatory solution of system (1.1). If $x(t) \to c$, then $y(t) \to 0$ as $t \to \infty$ for $0 < |c| < \infty$.*

According to the asymptotic behaviors of nonoscillatory solutions of system (1.1), M^+ and M^- can be divided into the following subclasses:

$$M^+_{B,B} = \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = d\right\},$$
$$M^+_{B,0} = \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = 0\right\},$$
$$M^+_{\infty,B} = \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \lim_{t\to\infty} |y(t)| = d\right\},$$

$$M^+_{\infty,0} = \left\{(x,y) \in M^+ : \lim_{t\to\infty} |x(t)| = \infty,\ \lim_{t\to\infty} |y(t)| = 0\right\},$$
$$M^-_{0,B} = \left\{(x,y) \in M^- : \lim_{t\to\infty} |x(t)| = 0,\ \lim_{t\to\infty} |y(t)| = d\right\},$$
$$M^-_{B,B} = \left\{(x,y) \in M^- : \lim_{t\to\infty} |x(t)| = c,\ \lim_{t\to\infty} |y(t)| = d\right\},$$
$$M^-_{0,\infty} = \left\{(x,y) \in M^- : \lim_{t\to\infty} |x(t)| = 0,\ \lim_{t\to\infty} |y(t)| = \infty\right\},$$
$$M^-_{B,\infty} = \left\{(x,y) \in M^- : \lim_{t\to\infty} |x(t)| = c,\ \lim_{t\to\infty} |y(t)| = \infty\right\},$$

where $0 < c < \infty$ and $0 < d < \infty$.

2. Existence of Nonoscillatory Solutions of (1.1) in M^+

We use the following improper integrals to classify nonoscillatory solutions of system (1.1).

$$Y_1 = \int_{t_0}^{\infty} p(t) f\left(k \int_t^{\infty} r(s)\Delta s\right)\Delta t, \quad Y_2 = \int_{t_0}^{\infty} r(t) g\left(l \int_{t_0}^{\sigma(t)} p(s)\Delta s\right)\Delta t,$$

$$Y_3 = \int_{t_0}^{\infty} p(t) f\left(c \int_{t_0}^{t} r(s)\Delta s\right)\Delta t, \qquad Y_4 = \int_{t_0}^{\infty} r(t) g\left(d \int_{\sigma(t)}^{\infty} p(s)\Delta s\right)\Delta t.$$

where k, l, c and d are nonzero constants.

2.1. The Case $P(t_0,\infty) = \infty$ and $R(t_0,\infty) < \infty$

Because Lemma 1.1 (c) and (g) eliminate $M^+_{B,B}$, we only consider the subclasses $M^+_{B,0}$, $M^+_{\infty,B}$ and $M^+_{\infty,0}$.

Theorem 2.1. *$M^+_{B,0} \neq \emptyset$ if and only if $Y_1 < \infty$ for some $k \neq 0$.*

Proof. Suppose that there exists $(x,y) \in M^+_{B,0} \neq \emptyset$ such that $x > 0$ eventually, $x(t) \to c > 0$ and $y(t) \to 0$ as $t \to \infty$. By the monotonicity of x and g, we have that there exists $k > 0$ and $t_1 \in \mathbb{T}$ such that $g(x^\sigma(t)) \geq k$ for $t \geq t_1$, $x^\sigma(t) = x(\sigma(t))$ Integrating the second equation of system (1.1) from t to ∞ gives

$$y(t) \geq k \int_t^{\infty} r(s)\Delta s \quad \text{for } t \geq t_1. \tag{2.1}$$

Integrating the first equation of system (1.1) from t_1 to t, monotonicity of f and inequality (2.1) give us

$$x(t) \geq x(t_1) + \int_{t_1}^{t} p(s) f\left(k \int_s^{\infty} r(u)\Delta u\right)\Delta s.$$

As $t \to \infty$, the assertion follows.

Conversely, suppose that $Y_1 < \infty$. Choose $t_1 \geq t_0$ and $k > 0$

$$\int_{t_0}^{\infty} p(t) f\left(k \int_t^{\infty} r(s)\Delta s\right)\Delta t < \frac{c_1}{2} \tag{2.2}$$

where $k = g(c_1)$. Let X be the set of all bounded, continuous, real valued functions with the norm $\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} \{|x(t)|\}$. It is clear that X is a Banach Space, see [5]. Define a subset Ω of X such that

$$\Omega := \left\{ x \in X : \ \frac{c_1}{2} \leq x(t) \leq c_1, \quad t \geq t_1 \right\}.$$

It is clear that Ω is closed, bounded and convex. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = c_1 - \int_t^\infty p(s) f\left(\int_s^\infty r(\tau) g\left(x^\sigma(\tau)\right) \Delta\tau \right) \Delta s \quad \text{for } t \geq t_1. \tag{2.3}$$

By (2.2), we have $F : \Omega \to \Omega$. We also need to show that F is continuous on Ω. So for $x_n \in \Omega$ that converges to $x \in \Omega$, one can show $\|(Fx_n)(t) - (Fx)(t)\| \to 0$ by the Lebesgue dominated convergence theorem, which implies the continuity of F on Ω. Furthermore, since

$$0 \leq -\left[F(x)(t)\right]^\Delta = p(t) f\left(\int_t^\infty r(\tau) g\left(x^\sigma(\tau)\right) \Delta\tau \right) < \infty,$$

it is shown that F is equibounded and equicontinuous. Therefore by the Schauder fixed point theorem, there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. So as $t \to \infty$, $\bar{x}(t) \to c_1$. We also get

$$\bar{x}^\Delta(t) = p(t) f\left(\int_t^\infty r(\tau) g(\bar{x}^\sigma(\tau)) \Delta\tau \right).$$

Setting $\bar{y}(t) = \int_t^\infty r(\tau) g(\bar{x}^\sigma(\tau))\Delta\tau$ and taking the limit as $t \to \infty$, it follows that $M_{B,0}^+ \neq \emptyset$. □

Theorem 2.2. *$M_{\infty,B}^+ \neq \emptyset$ if and only if $Y_2 < \infty$ for some $l \neq 0$.*

Proof. The sufficiency can be proven very similar to Theorem 2.1. So let us suppose that $Y_2 < \infty$. Then there exist $t_1 \geq t_0$ and $l > 0$ such that

$$\int_{t_1}^\infty r(s) g\left(l \int_{t_1}^{\sigma(s)} p(\tau)\Delta\tau \right) \Delta s < \frac{d_1}{2}, \tag{2.4}$$

where $l = f(d_1)$. Let X be the partially ordered Banach Space of all real-valued continuous functions with the norm $\sup x = \sup_{t>t_1} \frac{|x(t)|}{P(t_1,t)}$ and the usual pointwise ordering $\leq$. Define a subset Ω of X as follows:

$$\Omega := \left\{ x \in X : \ f\left(\frac{d_1}{2}\right) P(t_1,t) \leq x(t) \leq f(d_1) P(t_1,t) \ \text{ for } \ t > t_1 \right\}.$$

It is easy to show that $\inf B \in \Omega$, and $\sup B \in \Omega$ for every subset B. Define an operator $T : \Omega \to \ X$ as

$$(Tx)(t) = \int_{t_1}^t p(s) f\left(\frac{d_1}{2} + \int_s^\infty r(\tau) g\left(x^\sigma(\tau)\right) \Delta\tau \right) \Delta s. \tag{2.5}$$

So we have that $T : \Omega \to \Omega$ is an increasing mapping for $t \geq t_1$.

Hence, by the Knaster fixed point theorem, we have that there exists $\bar{x} \in \Omega$ such that $\bar{x} = T(\bar{x})$. By taking the derivative of $\bar{x}$, we obtain

$$\bar{x}^{\Delta}(t) = p(t) f\left(\frac{d_1}{2} + \int_t^{\infty} r(\tau) g(\bar{x}^{\sigma}(\tau))\Delta\tau\right).$$

And setting $\bar{y}(t) = \frac{d_1}{2} + \int_t^{\infty} r(\tau) g(\bar{x}^{\sigma}(\tau))\Delta\tau$ gives us $\bar{x}(t) \to \infty$ and $\bar{y}(t) \to \frac{d_1}{2}$ as $t \to \infty$, i.e., $M_{\infty,B}^{+} \neq \emptyset$. $\square$

Theorem 2.3. *If $Y_1 = \infty$ and $Y_2 < \infty$ for some $k, l \neq 0$, then $M_{\infty,0}^{+} \neq \emptyset$.*

Proof. Suppose that $Y_1 = \infty$ and $Y_2 < \infty$. Since $Y_2 < \infty$ and $P(t_0, \infty) = \infty$, we can choose $t_1 \geq t_0$ so large that

$$\int_{t_1}^{\infty} r(t) g\left(l \int_{t_1}^{\sigma(t)} p(s)\Delta s\right)\Delta t \leq 1 \tag{2.6}$$

and

$$P(t_1, \sigma(t)) \geq 1, \quad t \geq t_1 \tag{2.7}$$

where $l = 1 + f(1) > 0$. Let X be the Fréchet Space of all continuous functions on $[t_1, \infty)_{\mathbb{T}}$ endowed with the topology of uniform convergence on compact subintervals of $[t_1, \infty)_{\mathbb{T}}$. Set

$$\Omega := \left\{x \in X : \quad 1 \leq x(t) \leq 1 + f(1) \int_{t_1}^{t} p(s)\Delta s \text{ for } t \geq t_1\right\}$$

and define an operator $T : \Omega \to X$ by

$$(Tx)(t) = 1 + \int_{t_1}^{t} p(s) f\left(\int_s^{\infty} r(\tau) g\left(x^{\sigma}(\tau)\right)\Delta\tau\right)\Delta s. \tag{2.8}$$

Then there exists $t_2 \geq t_1$ such that we obtain for $t \geq t_2$

$$\begin{aligned}
1 \leq (Tx)(t) &\leq 1 + \int_{t_1}^{t} p(s) f\left(\int_s^{\infty} r(\tau) g\left(1 + f(1)\int_{t_1}^{\sigma(\tau)} p(u)\Delta u\right)\Delta\tau\right)\Delta s \\
&\leq 1 + \int_{t_1}^{t} p(s) f\left(\int_s^{\infty} r(\tau) g\left(\int_{t_1}^{\sigma(\tau)} p(u)\Delta u + f(1)\int_{t_1}^{\sigma(\tau)} p(u)\Delta u\right)\Delta\tau\right)\Delta s \\
&\leq 1 + \int_{t_1}^{t} p(s) f\left(\int_s^{\infty} r(\tau) g\left((1 + f(1))\int_{t_1}^{\sigma(\tau)} p(u)\Delta u\right)\Delta\tau\right)\Delta s \\
&\leq 1 + f(1)\int_{t_1}^{t} p(s)\Delta s,
\end{aligned}$$

where we use (2.6) and (2.7). This implies $T : \Omega \to \Omega$. Next, we show that T is continuous on Ω. Let x_n be a sequence in Ω such that $x_n \to x \in \Omega = \bar{\Omega}$. Then, we have

$$\begin{aligned}
&|(Tx_n - Tx)(t)| \\
&\quad \leq \int_{t_1}^{t} p(s)\left|f\left(\int_s^{\infty} r(\tau) g\left(x^{\sigma}(\tau)\right)\Delta\tau\right) - f\left(\int_s^{\infty} r(\tau) g\left(x_n^{\sigma}(\tau)\right)\Delta\tau\right)\right|
\end{aligned}$$

for $t \geq t_2$. Hence, the Lebesque Dominated Convergence theorem and the continuity of g give $\|(Tx_n)-(Tx)\| \to 0$ as $n \to \infty$, i.e., T is continuous on Ω. Also by the similar discussion in Theorem 2.1, we have that T is equibounded and equicontinuous. Then by Tychonoff Fixed Point Theorem, there exists $\bar{x} \in \Omega$ such that $\bar{x} = T\bar{x}$. Therefore, by setting $\bar{y}(t) = \int_t^\infty r(\tau) g\left(\bar{x}^\sigma(\tau)\right)\Delta\tau$ and taking the limit of (2.8) and $\bar{y}(t)$ as $t \to \infty$, we have that $M^+_{\infty,0} \neq \emptyset$. □

2.2. The Case $P(t_0, \infty) < \infty$ and $R(t_0, \infty) < \infty$

By Lemma 1.1 (e), and (f), we have that the component functions x and y has to converge a finite number. So the subclasses $M^+_{\infty,0} = M^+_{\infty,B} = \emptyset$. In view of this information, we have that a nonoscillatory solution in M^+ might belong to $M^+_{B,B}$ or $M^+_{B,0}$.

Necessary and sufficient condition to have a nonoscillatory solution of system (1.1) in $M^+_{B,B}$ or $M^+_{B,0}$ is $Y_1 < \infty$. The proof can be shown very similar to Theorem 2.2 in [11] and Theorem 2.1, respectively.

3. Existence of Nonoscillatory Solutions of (1.1) in M^-

3.1. The Case $P(t_0, \infty) < \infty$ and $R(t_0, \infty) = \infty$

Because all nonoscillatory solutions belong to M^- by Lemma 1.1 (d), we only focus on M^- in this subsection.

Theorem 3.1. *$M^-_{B,\infty} \neq \emptyset$ if and only if $Y_3 < \infty$, where $c \neq 0$ and f is an odd function.*

Proof. The suffiency can be proven very similar to Theorem 8 in [10]. So it is omitted. Conversely, suppose $Y_3 < \infty$. Then choose $t_1 \geq t_0$ and $c > 0$ such that

$$\int_{t_1}^{\infty} p(s) f\left(c\int_{t_1}^{s} r(\tau)\Delta\tau\right)\Delta s < \frac{d}{2},$$

where $c = g(d)$. Let X be the partially ordered Banach space of all real valued continuous functions endowed with the norm $\|x\| = \sup_{t \geq t_1} |x(t)|$ and the usual pointwise ordering $\leq$. Define a subset Ω of X and an operator $F := \Omega \to X$ as

$$\Omega := \left\{x \in X : \quad \frac{d}{2} \leq x(t) \leq d, \quad t \geq t_1\right\}$$

and

$$(Fx)(t) = \frac{d}{2} + \int_t^\infty p(s) f\left(\int_{t_1}^{s} r(\tau) g(x^\sigma(\tau))\Delta\tau\right)\Delta s, \quad t \geq t_1. \tag{3.1}$$

It is easy to show that F is an increasing mapping into itself and $(\Omega, \leq)$ is a complete space. Then by the Knaster fixed point theorem there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. Then by taking limit of (3.1) as $t \to \infty$, using the fact that f is an odd function and setting $\bar{y}(t) = -\int_{t_1}^{s} r(\tau) g(\bar{x}^\sigma(\tau))\Delta\tau$, we have that the assertion follows. □

Remark 3.2. We refer the reader [10, Theorem 2.4] for the proof of that if $Y_1 < \infty$ and $Y_4 = \infty$, then $M^-_{0,\infty} \neq \emptyset$, where f is an odd function.

3.2. Emden–Fowler Systems

In this section, we reduce system (1.1) into

$$\begin{cases} x^{\Delta}(t) = p(t)\,|y(t)|^{\gamma}\, sgny(t) \\ y^{\Delta}(t) = -r(t)\,|x^{\sigma}(t)|^{\beta}\, sgnx^{\sigma}(t) \end{cases} \tag{3.2}$$

by substituting $f(z) = |z|^{\gamma-1}z$, $g(z) = |z|^{\beta-1}z$ in (1.1), where $\gamma, \beta > 0$ and $p, r \in C_{rd}\left([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+\right)$, which is known as a system of Emden–Fowler dynamic equations on time scales. We use the reciprocal principle in order to show the existence of nonoscillatory solutions of system (3.2) in M^-, where μ has to be delta-differentiable on $\mathbb{T}$, where $\mu(t) = \sigma(t) - t$. It is clear that if (x, y) is a solution of system (3.2), then (u, v) is a solution of

$$\begin{cases} u^{\Delta}(t) = m(t)\,|v(t)|^{\beta}\, sgnv(t) \\ v^{\Delta}(t) = -n(t)\,|u^{\sigma}(t)|^{\gamma}\, sgnu^{\sigma}(t), \end{cases} \tag{3.3}$$

where $u = y$, $v = -x^{\sigma}$, $m(t) = r(t)$ and $n(t) = (1 + \mu^{\Delta}(t))p^{\sigma}(t)$. One can easily show that $(x, y) \in M^-$ if and only if $(u, v) \in M^+$. Therefore, we have the followings:

$$\begin{aligned} (x,y) \in M^-_{B,\infty} &\quad \text{iff } (u,v) \in M^+_{\infty,B}, \\ (x,y) \in M^-_{0,\infty} &\quad \text{iff } (u,v) \in M^+_{\infty,0}, \\ (x,y) \in M^-_{0,B} &\quad \text{iff } (u,v) \in M^+_{B,0}, \\ (x,y) \in M^-_{B,B} &\quad \text{iff } (u,v) \in M^+_{B,B}. \end{aligned} \tag{3.4}$$

Since we have the results for system (1.1) in $M^-_{B,\infty}$ and $M^-_{0,\infty}$ in Sect. 3.1, it is enough to show the existence of nonoscillatory solutions of (3.2) in $M^-_{B,B}$ and $M^-_{0,B}$. In order to do that, we need to modify the integrals $P(t_0,\infty), R(t_0,\infty)$ and Y_1. Therefore, our new improper integrals turn out to be as follows:

$$\begin{aligned} \overline{Y}_1 &= \int_{t_0}^{\infty} r(t) \left(\int_t^{\infty} (1 + \mu^{\Delta}(s))p^{\sigma}(s)\Delta s \right)^{\beta} \Delta t, \\ \overline{P}(t_0,\infty) &= \int_{t_0}^{\infty} m(t)\Delta t = \int_{t_0}^{\infty} r(t)\Delta t, \\ \overline{R}(t_0,\infty) &= \int_{t_0}^{\infty} n(t)\Delta t = \int_{t_0}^{\infty} (1 + \mu^{\Delta}(t))p^{\sigma}(t)\Delta t. \end{aligned}$$

Recall from Sect. 2 that there exists a nonoscillatory solution in $M^+_{B,0}$ under the cases $P(t_0,\infty) = \infty, R(t_0,\infty) < \infty$ (or $P(t_0,\infty) = \infty, R(t_0,\infty) < \infty$) and $Y_1 < \infty$. Similarly $M^+_{B,B}$, we have $P(t_0,\infty) < \infty, R(t_0,\infty) < \infty$ and $Y_1 < \infty$. By using the integrals above and the relation (3.4), one can easily prove the following theorem.

Theorem 3.3. (a) *Suppose that* $\overline{P}(t_0,\infty) = \infty$ *and* $\overline{R}(t_0,\infty) < \infty$. (*Or* $\overline{P}(t_0,\infty) < \infty$, $\overline{R}(t_0,\infty) < \infty$.) *Then* $M^-_{0,B} \neq \emptyset$ *if and only if* $\overline{Y}_1 < \infty$.

(b) *Suppose that* $\overline{P}(t_0,\infty) < \infty$, $\overline{R}(t_0,\infty) < \infty$. *Then* $M^-_{B,B} \neq \emptyset$ *if and only if* $\overline{Y}_1 < \infty$.

4. Conclusion and Some Examples

Example. Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $\gamma = \frac{1}{2}$, $\beta = \frac{1}{3}$, $p(t) = \dfrac{t^{\frac{1}{2}}}{2(t+1)(t+2)(3t-1)^{\frac{1}{2}}}$, $r(t) = \dfrac{(t+1)^{\frac{1}{3}}}{2^{\frac{2}{3}}t^2(4t+5)^{\frac{1}{3}}}$ in system (3.2). We show that there exists a nonoscillatory solution $(x,y) \in M_{B,B}^-$. In order to do that, we first need to show $\overline{P}(1,\infty) < \infty$ and $\overline{R}(1,\infty) < \infty$. Indeed,

$$\overline{P}(1,T) = \int_1^T r(t)\Delta t = \sum_{t\in[1,T)_{2^{\mathbb{N}_0}}} \frac{(t+1)^{\frac{1}{3}}t}{2^{\frac{2}{3}}t^2(4t+5)^{\frac{1}{3}}} \leq \sum_{t\in[1,T)_{2^{\mathbb{N}_0}}} \frac{(t+1)^{\frac{1}{3}}}{t}.$$

Therefore, as $T \to \infty$ we have

$$\sum_{n=0}^{\infty} \frac{(2^n+1)^{\frac{1}{3}}}{2^n} < \infty$$

by the Ratio test, i.e., $\overline{P}(1,\infty) < \infty$. Similarly,

$$\overline{R}(1,T) = \sum_{t\in[1,T)_{2^{\mathbb{N}_0}}} \frac{(2t)^{\frac{1}{2}} \cdot t}{(2t+1)(2t+2)(6t-1)^{\frac{1}{2}}} \leq \sum_{t\in[1,T)_{2^{\mathbb{N}_0}}} \frac{t^{\frac{3}{2}}}{(2t+1)(t+1)}.$$

So, as $T \to \infty$, we have that $\overline{R}(1,\infty) < \infty$. We also have $\overline{Y}_1 < \infty$ since $\overline{P}(1,\infty) < \infty$ and $\overline{R}(1,\infty) < \infty$. One can show that $(2 + \frac{1}{t+2}, -3 + \frac{1}{t})$ is a nonoscillatory solution of

$$\begin{cases} x^{\Delta}(t) = \dfrac{t^{\frac{1}{2}}}{2(t+1)(t+2)(3t-1)^{\frac{1}{2}}} \left|y(t)\right|^{\frac{1}{2}} sgn y(t) \\ y^{\Delta}(t) = -\dfrac{(t+1)^{\frac{1}{3}}}{2^{\frac{2}{3}}t^2(4t+5)^{\frac{1}{3}}} \left|x^{\sigma}(t)\right|^{\frac{1}{3}} sgn x^{\sigma}(t) \end{cases}$$

such that $x(t) \to 2$ and $y(t) \to -3$ as $t \to \infty$, i.e., $M_{B,B}^- \neq \emptyset$ by Theorem 3.3.

Example. Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $f(z) = z^{\frac{1}{3}}$, $g(z) = z^{\frac{1}{5}}$, $p(t) = \dfrac{3}{4t^{\frac{10}{3}}}$, $r(t) = \left(\dfrac{4t^2}{4t^2+1}\right)^{\frac{1}{5}}$ in system (1.1). We show that there exists a nonoscillatory solution $(x,y) \in M_{B,\infty}^-$. To do that, we first need to show $P(1,\infty) < \infty$ and $R(1,\infty) = \infty$.

$$P(1,T) = \int_1^T p(t)\Delta t = \sum_{t\in[1,T)_{2^{\mathbb{N}_0}}} \frac{3}{4t^{\frac{7}{3}}}.$$

Therefore, as $T \to \infty$ we have that $P(1,\infty) < \infty$ by the Ratio test. Similarly,

$$R(1,T) = \sum_{t\in[1,T)_{2^{\mathbb{N}_0}}} \left(\frac{4t^2}{4t^2+1}\right)^{\frac{1}{5}} t.$$

So, as $T \to \infty$, we have that $R(1,\infty) = \infty$ by the limit divergence test. Now we need to show $Y_3 < \infty$.

$$\int_1^s r(\tau)\Delta\tau \le \sum_{\tau\in[1,s)_{2^{\mathbb{N}_0}}} \tau = s-1. \tag{4.1}$$

Therefore, by (4.1) and by letting $c = 1$, we have

$$\begin{aligned}\int_1^T p(s) f\left(c\int_1^s r(\tau)\Delta\tau\right)\Delta s &\le \int_1^T \frac{3}{4s^{\frac{10}{3}}}(s-1)^{\frac{1}{3}}\Delta s\\ &= \sum_{s\in[1,T)_{2^{\mathbb{N}_0}}} \frac{3(s-1)^{\frac{1}{3}}s}{4s^{\frac{10}{3}}} \le \sum_{s\in[1,T)_{2^{\mathbb{N}_0}}} \frac{1}{s^2}\end{aligned}$$

Hence, we have $Y_3 < \infty$ as $T \to \infty$ by the Comparison theorem and geometric series. It can be shown that $(1+\frac{1}{t^2}, -t)$ is a nonoscillatory solution of

$$\begin{cases} x^{\Delta}(t) = \dfrac{3}{4t^{\frac{10}{3}}}(y(t))^{\frac{1}{3}}\\ y^{\Delta}(t) = -\left(\dfrac{4t^2}{4t^2+1}\right)^{\frac{1}{5}}(x(2t))^{\frac{1}{5}}\end{cases}$$

such that $x(t) \to 1$ and $y(t) \to -\infty$ as $t \to \infty$, i.e., $M^-_{B,\infty} \neq \emptyset$ by Theorem 3.1.

Finally, we summarize the main results in following Tables 1 and 2.

Table 1. Classification for (1.1) in M^+ and M^-

$M^+_{B,0}$	$\neq \emptyset$	$P(t_0,\infty) = \infty$ and $R(t_0,\infty) < \infty$	$Y_1 < \infty$
		$P(t_0,\infty) < \infty$ and $R(t_0,\infty) < \infty$	
$M^+_{\infty,B}$	$\neq \emptyset$	$P(t_0,\infty) = \infty$ and $R(t_0,\infty) < \infty$	$Y_2 < \infty$
$M^+_{\infty,0}$	$\neq \emptyset$	$P(t_0,\infty) = \infty$ and $R(t_0,\infty) < \infty$	$Y_1 = \infty$ and $Y_2 < \infty$
$M^+_{B,B}$	$\neq \emptyset$	$P(t_0,\infty) < \infty$ and $R(t_0,\infty) < \infty$	$Y_1 < \infty$

Table 2. Classification for (3.2) in M^-

$M^-_{0,B}$	$\neq \emptyset$	$\overline{P}(t_0,\infty) = \infty$ and $\overline{R}(t_0,\infty) < \infty$	$\overline{Y}_1 < \infty$
		$\overline{P}(t_0,\infty) < \infty$ and $\overline{R}(t_0,\infty) < \infty$	
$M^-_{B,B}$	$\neq \emptyset$	$\overline{P}(t_0,\infty) < \infty$ and $\overline{R}(t_0,\infty) < \infty$	$\overline{Y}_1 < \infty$

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