Mathematical
Inequalities
& Applications
Volume 20, Number 2 (2017), 377–387 doi:10.7153/mia-20-26

CLASSIFICATION SCHEMES OF NONOSCILLATORY SOLUTIONS FOR TWO–DIMENSIONAL TIME SCALE SYSTEMS

ÖZKAN ÖZTÜRK

(*Communicated by M. Bohner*)

Abstract. Asymptotic properties of solutions for nonlinear systems are significant in order to obtain enough information about the behavior of systems. We deal with a two dimensional time scale nonlinear system and show the (non)existence of nonoscillatory solutions by using most well – known fixed point theorems. We also provide several examples whose solutions are known explicitly.

In this paper, we consider the system

$$\begin{cases} x^{\Delta}(t) = p(t)f(y(t)) \\ y^{\Delta}(t) = r(t)g(x(t)), \end{cases} \tag{1}$$

where $p,r \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ and f and g are nondecreasing functions such that $uf(u) > 0$ and $ug(u) > 0$ for $u \neq 0$. A time scale $\mathbb{T}$, a nonempty closed subset of real numbers, is introduced by Stefan Hilger in his PhD thesis in 1988 in order to harmonize discrete and continuous analyses to combine them in one comprehensive theory and eliminate obscurity from both. In 2001 and 2003, the time scale theory was published in a series of two books by Bohner and Peterson, see [2] and [3]. Throughout this paper, we assume that $\mathbb{T}$ is unbounded above and whenever we write $t \geqslant t_1$, we mean $t \in [t_1,\infty)_{\mathbb{T}} := [t_1,\infty) \cap \mathbb{T}$. We call (x,y) a *proper solution* if it is defined on $[t_0,\infty)_{\mathbb{T}}$ and $\sup\{|x(s)|,|y(s)| : s \in [t,\infty)_{\mathbb{T}}\} > 0$ for $t \geqslant t_0$. A solution (x,y) of (1) is said to be nonoscillatory if the component functions x and y are both nonoscillatory, i.e., either eventually positive or eventually negative. Otherwise it is said to be oscillatory. Some of the examples for the most known time scales are the set of real numbers $\mathbb{R}$, the set of integers $\mathbb{Z}$ and $q^{\mathbb{N}_0}$, where $q > 1$ and $\mathbb{N}_0 = \{0,1,2\dots\}$.

For example, if $\mathbb{T} = \mathbb{R}$ equation (1) reduces to the system of first order differential equations

$$\begin{cases} x'(t) = p(t)f(y(t)) \\ y'(t) = r(t)g(x(t)), \end{cases}$$

see [7].

Mathematics subject classification (2010): 34N05, 39A10, 39A13.
Keywords and phrases: Time scales, two dimensional nonlinear time scale system, integral inequalities, nonoscillation, oscillation theory.

Recently, Öztürk and Akın have studied a class of two dimensional time scale delay system

$$\begin{cases} x^{\Delta}(t) = p(t)f(y(t)) \\ y^{\Delta}(t) = -r(t)g(x(\tau(t))), \end{cases} \tag{2}$$

where $\tau(t) : \mathbb{T} \to \mathbb{T}$ is a delay function such that $\tau(t) \leqslant t$ and $\tau(t) \to \infty$ as $t \to \infty$ and $r > 0$, see [11]. The case $\tau(t) = t$ in system (2) and other variations of systems (1) and (2) are also studied in [8], [9] and [10].

Let M be the set of all nonoscillatory solutions of system (1). One can easily show that any nonoscillatory solution (x,y) of system (1) belongs to one of the following classes:

$$M^{+} := \{(x,y) \in M : \ xy > 0 \text{ eventually}\}$$
$$M^{-} := \{(x,y) \in M : \ xy < 0 \text{ eventually}\}.$$

In this paper, we only focus on M^{+} by assuming $x > 0$ eventually without loss of generality. Proofs can also be shown similarly when $x < 0$ eventually. The setup of this paper is as follows: In Section 1, we give preliminaries that play important roles in further sections. In Section 2, we show the existence and asymptotic properties of nonoscillatory solutions of system (1) by using certain improper integrals and fixed point theorems. In Section 3, we provide a few examples in order to underline our main results. Finally, we finish the paper by giving a conclusion and open problems .

Let (x,y) be a solution of system (1). Then it is easy to show that the component functions x and y are themselves nonoscillatory, see eg. [1].

For convenience, let us set

$$P(t_0,t) = \int_{t_0}^{t} p(s)\Delta s \quad \text{and} \quad R(t_0,t) = \int_{t_0}^{t} r(s)\Delta s$$

and consider the following four cases for $t_0 \in \mathbb{T}$:

(i) $P(t_0,\infty) = \infty$ and $R(t_0,\infty) = \infty$,

(ii) $P(t_0,\infty) = \infty$ and $R(t_0,\infty) < \infty$,

(iii) $P(t_0,\infty) < \infty$ and $R(t_0,\infty) < \infty$,

(iv) $P(t_0,\infty) < \infty$ and $R(t_0,\infty) = \infty$.

Suppose that (x,y) is a positive nonoscillatory solution of system (1). Then by the first and second equations of system (1), we have that $x^{\Delta} > 0$ and $y^{\Delta} > 0$ eventually. So we have that $x \to c$ or $x \to \infty$ and $y \to c$ or $y \to \infty$ for $0 < c < \infty$ and $0 < d < \infty$. For $x < 0$, we have the similar limit behaviors. So in the light of this information, we have the following sub – classes of M^{+}.

$$M^{+}_{B,B} = \left\{(x,y) \in M^{+} : \lim_{t\to\infty} |x(t)| = c, \ \lim_{t\to\infty} |y(t)| = d\right\},$$
$$M^{+}_{B,\infty} = \left\{(x,y) \in M^{+} : \lim_{t\to\infty} |x(t)| = c, \ \lim_{t\to\infty} |y(t)| = \infty\right\},$$
$$M^{+}_{\infty,B} = \left\{(x,y) \in M^{+} : \lim_{t\to\infty} |x(t)| = \infty, \ \lim_{t\to\infty} |y(t)| = d\right\},$$
$$M^{+}_{\infty,\infty} = \left\{(x,y) \in M^{+} : \lim_{t\to\infty} |x(t)| = \infty, \ \lim_{t\to\infty} |y(t)| = \infty\right\}.$$

Now we show that some of the sub – classes mentioned above can be eliminated under the case (i). So suppose that $P(t_0,\infty)=\infty$ and $R(t_0,\infty)=\infty$ and that (x,y) is a nonoscillatory solution in M^+. Without loss of generality assume that $x>0$ and $y>0$ eventually. Integrating the first and second equations of system (1) from t_0 to t and the monotonicity of f and g give us

$$x(t)\geqslant x(t_0)+f(y(t_0))\int_{t_0}^{t}p(s)\Delta s$$

and

$$y(t)\geqslant y(t_0)+g(x(t_0))\int_{t_0}^{t}r(s)\Delta s,\quad t\geqslant t_0.$$

Hence, as $t\to\infty$, we have that $x(t)\to\infty$ and $y(t)\to\infty$. In view of this information, one can prove the following theorem.

THEOREM 1. *Suppose that $P(t_0,\infty)=\infty$ and $R(t_0,\infty)=\infty$. Then any nonoscillatory solution of system (1) belongs to $M^+_{\infty,\infty}$.*

In next sections, we show that some of the subclasses above might be empty according to the convergence/divergence of $P(t_0,\infty)$ and $R(t_0,\infty)$ by using the following fixed point theorems.

THEOREM 2. (Schauder's Fixed Point Theorem) [12, Theorem 2.A] *Let M be a nonempty, closed, bounded, convex subset of a Banach space X, and suppose that $T:M\to M$ is a compact operator. Then, T has a fixed point.*

The following theorem is the alternate version of the Schauder's fixed point theorem, see [12].

COROLLARY 1. *Let M be a nonempty, compact, convex subset of a Banach space X, and suppose that $T:M\to M$ is a continuous operator. Then, T has a fixed point.*

THEOREM 3. (Knaster Fixed Point Theorem) [6] *If $(M,\leqslant)$ is a complete lattice and $T:M\to M$ is order-preserving (also called monotone or isotone), then T has a fixed point. In fact, the set of fixed points of T is a complete lattice.*

1. Existence of nonoscillatory solutions of (1) in M^+

In this section, we obtain the (non)existence criteria for nonoscillatory solutions of system (1) by using the fixed point theorems given above via the following improper integrals:

$$Y_1=\int_{t_0}^{\infty}p(t)f\left(l\int_{t_0}^{t}r(s)\Delta s\right)\Delta t,$$
$$Y_2=\int_{t_0}^{\infty}r(t)g\left(k\int_{t_0}^{t}p(s)\Delta s\right)\Delta t,$$

where l and k are nonzero constants.

1.1. The case $P(t_0,\infty)=\infty$ and $R(t_0,\infty)<\infty$

In this subsection, we give the first classification schemes of nonoscillatory solutions of (1) in M^+ under the case $P(t_0,\infty)=\infty$ and $R(t_0,\infty)<\infty$. Let (x,y) be a nonoscillatory solution of system (1) such that $x>0$ and $y>0$ eventually. Then by integrating the first equation of system (1) from t_0 to t, monotonicity of f and y, we have that there exists $k>0$

$$x(t)\geqslant x(t_0)+k\int_{t_0}^{t} p(s)\Delta s, \quad t_0\in\mathbb{T}. \tag{3}$$

Then by taking the limit of (3) as $t\to\infty$, we have $x(t)\to\infty$. So in view of this information, we have the following lemma.

LEMMA 1. *For $0<c,d<\infty$, any nonoscillatory solution in M^+ must belong to $M^+_{\infty,B}$, or $M^+_{\infty,\infty}$.*

It is not easy to give the sufficient conditions for the existence of nonoscillatory solutions in $M^+_{\infty,\infty}$. Therefore we have the following theorem.

THEOREM 4. *$M^+_{\infty,B}\neq\emptyset$ if and only if $Y_2<\infty$.*

Proof. Suppose that there exists a solution in $M^+_{\infty,B}$ such that $x(t)>0$, $y(t)>0$ for $t\geqslant t_0$, $x(t)\to\infty$ and $y(t)\to d$ as $t\to\infty$ for $d>0$. Since y is eventually increasing, there exist $k>0$ and $t_1\geqslant t_0$ such that $f(y(t))\geqslant k$ for $t\geqslant t_1$. Integrating the first equation from t_1 to t and the monotonicity of f yield us

$$x(t)=x(t_1)+\int_{t_1}^{t} p(s)f(y(s))\Delta s\geqslant k\int_{t_1}^{t} p(s)\Delta s, \quad t\geqslant t_1. \tag{4}$$

Integrating the second equation from t_1 to t, the monotonicity of g and (4) give us

$$y(t)=y(t_1)+\int_{t_1}^{t} r(s)g(x(s))\Delta s\geqslant \int_{t_1}^{t} r(s)g\left(k\int_{t_1}^{s} p(u)\Delta u\right)\Delta s, \quad t\geqslant t_1. \tag{5}$$

So as $t\to\infty$, we have that $Y_2<\infty$ holds.

Conversely, suppose that $Y_2<\infty$. Then there exist a large $t_1\geqslant t_0$ and $k>0$ such that

$$\int_{t_1}^{\infty} r(t)g\left(k\int_{t_1}^{t} p(s)\Delta s\right)\Delta t<\frac{c}{2}, \tag{6}$$

where $k=f(c)$. Let X be the set of all bounded and continuous real valued functions $y(t)$ on $[t_1,\infty)_{\mathbb{T}}$ with the supremum norm $\sup_{t\geqslant t_1}|y(t)|$. Then X is a Banach space, see [5]. Let us define a subset Ω of X such that

$$\Omega:=\left\{y(t)\in X:\ \frac{c}{2}\leqslant y(t)\leqslant c,\ t\geqslant t_1\right\}.$$

It is easy to show that Ω is closed, bounded and convex subset of X. Define an operator $T:\Omega\to X$ such that

$$(Ty)(t) = c - \int_t^\infty r(s)g\left(\int_{t_1}^s p(u)f(y(u))\Delta u\right)\Delta s. \tag{7}$$

First, let us show $T:\Omega\to\Omega$. Indeed,

$$c \geqslant (Ty)(t) \geqslant c - \int_t^\infty r(s)g\left(\int_{t_1}^s p(u)f(c)\Delta u\right)\Delta s \geqslant \frac{c}{2}$$

by the definition of Ω and (6). Next, we need to show T is continuous on Ω. So let y_n be a sequence in Ω such that $\|y_n - y\|\to 0$, where $y\in\Omega$. Then

$$|(Ty_n)(t)-(Ty)(t)| \leqslant \int_t^\infty r(s)\left|g\left(\int_{t_1}^s p(u)f(y_n(u))\Delta u\right) - g\left(\int_{t_1}^s p(u)f(y(u))\Delta u\right)\right|\Delta s$$

Then by the Lebesgue dominated convergence theorem and by the continuity of f and g, we have that $\|Ty_n - Ty\|\to 0$ as $n\to\infty$, i.e., T is continuous. Finally, we show that $T\Omega$ is relatively compact, i.e., equibounded and equicontinuous. Since

$$0 < (Ty)^\Delta(t) = r(t)g\left(\int_{t_1}^t p(u)f(y(u))\Delta u\right) \leqslant r(t)g\left(k\int_{t_1}^t p(u)\Delta u\right) < \infty,$$

we have that Ty is relatively compact by the Arzelá – Ascoli and mean value theorems. Therefore, by the Schauder fixed point theorem, we have that there exists $\bar{y}\in\Omega$ such that $\bar{y} = T\bar{y}$. Then we have

$$\bar{y}^\Delta(t) = (T\bar{y})^\Delta(t) = r(t)g\left(\int_{t_1}^t p(u)f(\bar{y}(u))\Delta u\right), \quad t\geqslant t_1. \tag{8}$$

Setting $\bar{x}(t) = \int_{t_1}^t p(u)f(\bar{y}(u))\Delta u$ gives us $x^\Delta(t) = p(t)f(\bar{y}(t))$. So we have that $(\bar{x},\bar{y})$ is a nonoscillatory solution of system (1) such that $\bar{x}(t)\to\infty$ and $\bar{y}(t)\to c$ as$t\to\infty$, i.e., $M^+_{\infty,B}\neq\emptyset$. $\square$

1.2. The case $P(t_0,\infty)<\infty$ and $R(t_0,\infty)<\infty$

In this subsection, we show that the existence of nonoscillatory solutions of (1) can only be acquired in $M^+_{B,B}$ and $M^+_{\infty,\infty}$ for $P(t_0,\infty)<\infty$ and $R(t_0,\infty)<\infty$, i.e., $M^+_{B,\infty} = M^+_{\infty,B} = \emptyset$.

LEMMA 2. *Suppose $P(t_0,\infty)<\infty$ and $R(t_0,\infty)<\infty$ holds. Then $x(t)\to c$ if and only if $y(t)\to d$ as $t\to\infty$, where $0<|c|<\infty$ and $0<|d|<\infty$.*

Proof. Without loss of generality, let us assume that (x,y) is a nonoscillatory solution of system (1) such that $x>0$ and $y>0$ eventually and x has a finite limit.

Integrating the second equation of system (1) and monotonicity of x and g give us that there exists $k > 0$ such that

$$y(t) \leqslant y(t_0) + k \int_{t_0}^{t} r(s)\Delta s,$$

where $k = g(c)$. Then as $t \to \infty$, we have that y has a finite limit since $P(t_0, \infty) < \infty$. The other direction can be proven similarly. This proves the assertion. $\square$

THEOREM 5. *$M_{B,B}^{+} \neq \emptyset$ if and only if $Y_1 < \infty$.*

Proof. Suppose that there exists a nonoscillatory solution $(x,y) \in M_{B,B}^{+}$ such that $x > 0$, $y > 0$ eventually, $x(t) \to c$ and $y(t) \to d$ as $t \to \infty$ for $0 < c, d < \infty$. (The case $x < 0$ and $y < 0$ eventually can be shown similarly.) Integrating the second equation of system (1) from t_0 to t and monotonicity of x and g give us

$$y(t) \geqslant l \int_{t_0}^{t} r(s)\Delta s, \tag{9}$$

where $l = g(x(t_0))$. Now by integrating the first equation of system (1) from t_0 to t and by (9), we have

$$x(t) \geqslant \int_{t_0}^{t} p(s) f\left(l \int_{t_0}^{s} r(u)\Delta u\right) \Delta s.$$

Hence, we have $Y_1 < \infty$ as $t \to \infty$.

Conversely, suppose $Y_1 < \infty$ holds. Then choose $t_1 \geqslant t_0$ and $l > 0$ such that

$$\int_{t_1}^{\infty} p(t) f\left(l \int_{t_1}^{t} r(s)\Delta s\right) \Delta t < \frac{c}{2}, \tag{10}$$

where $l = g(c)$ and $t \geqslant t_1$. Let X be the Banach space of all bounded real-valued and continuous functions on $[t_0, \infty)_{\mathbb{T}}$ endowed with the norm $\sup\limits_{t \geqslant t_1} |x(t)|$ and usual pointwise ordering $\leqslant$. Define a subset Ω of X as

$$\Omega := \left\{ x \in X : \frac{c}{2} \leqslant x(t) \leqslant c \; t \geqslant t_1 \right\}$$

and an operator $T : \Omega \to X$ such that

$$(Tx)(t) = \frac{c}{2} + \int_{t_1}^{t} p(s) f\left(\int_{t_1}^{s} r(u) g(x(u))\Delta u\right) \Delta t, \quad t \geqslant t_1.$$

It is easy to show that $\inf B \in \Omega$ and $\sup B \in \Omega$ for any subset B of Ω, i.e., $(\Omega, \leqslant)$ is a complete lattice. First we need to show $T : \Omega \to \Omega$ is an increasing mapping. Indeed,

$$\frac{c}{2} \leqslant (Tx)(t) \leqslant \frac{c}{2} + \int_{t_1}^{t} p(s) f\left(g(c) \int_{t_1}^{s} r(u)\Delta u\right) \Delta t \leqslant c, \quad t \geqslant t_1,$$

i.e., $T : \Omega \to \Omega$. It can also be easily shown that T is an increasing mapping. Then by the Knaster fixed point theorem, there exists a function $\bar{x} \in \Omega$ such that $\bar{x} = T\bar{x}$. By taking the derivative of $T\bar{x}$, we have

$$(T\bar{x})^{\Delta}(t) = p(t)f\left(\int_{t_1}^{t} r(u)g(\bar{x}(u))\Delta u\right), \quad t \geqslant t_1.$$

Setting

$$\bar{y}(t) = \int_{t_1}^{t} r(u)g(\bar{x}(u))\Delta u$$

gives us $\bar{y}^{\Delta}(t) = r(t)g(\bar{x}(t))$ and $(\bar{x}, \bar{y})$ is a nonoscillatory solution of system (1) such that $\bar{x}$ and $\bar{y}$ have finite limits as $t \to \infty$. This completes the proof. □

REMARK 1. Note that if $P(t_0, \infty) < \infty$ and $R(t_0, \infty) < \infty$, then $Y_1 < \infty$. So Theorem 5 also holds for $P(t_0, \infty) < \infty$ and $R(t_0, \infty) < \infty$.

1.3. The case $P(t_0, \infty) < \infty$ and $R(t_0, \infty) = \infty$

In this subsection, we give the nonoscillation criteria in M^+ under the case $P(t_0, \infty) < \infty$ and $R(t_0, \infty) = \infty$. Therefore, we have the following lemma.

LEMMA 3. *Suppose that $R(t_0, \infty) = \infty$. Then any nonoscillatory solution in M^+ belongs to $M^+_{B,\infty}$ or $M^+_{\infty,\infty}$, i.e., $M^+_{B,B} = M^+_{\infty,B} = \emptyset$.*

Proof. Suppose that $R(t_0, \infty) = \infty$ and (x, y) is a nonoscillatory solution of system (1) in M^+ such that $x > 0$ and $y > 0$ eventually. Integrating the second equation of system (1) from t_0 to t gives us

$$y(t) \geqslant k \int_{t_0}^{t} r(s)\Delta s,$$

where $k = g(x(t_0))$. So as $t \to \infty$, we have that the limit of y is infinite, i.e., $M^+_{B,B} = M^+_{\infty,B} = \emptyset$. Therefore, the assertion follows. □

The following theorem shows us the (non)existence of nonoscillatory solutions in $M^+_{B,\infty}$. We skip the proof of the following theorem, since it is very similar to the proof of Theorem 4.

THEOREM 6. *$M^+_{B,\infty} \neq \emptyset$ if and only if $Y_1 < \infty$.*

2. Examples

In this section we pay our attention on several examples in order to highlight our main results.

EXAMPLE 1. Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $k = 1$, $f(z) = z^{\frac{1}{61}}$, $g(z) = z^{\frac{3}{5}}$, $p(t) = \left(\frac{t}{2t-1}\right)^{\frac{1}{61}}$, $r(t) = \frac{1}{2t^{\frac{13}{5}}}$, and $t = 2^n$ in system (1). We show that there exists a nonoscillatory solution in $M^{+}_{\infty,B}$. In order to do that, we need to show $P(t_0,\infty) = \infty$ and $R(t_0,\infty) < \infty$. Indeed,

$$P(1,T) = \int_1^T p(t)\Delta t = \sum_{t\in[1,T)_{2^{\mathbb{N}_0}}} \left(\frac{t}{2t-1}\right)^{\frac{1}{61}} \cdot t.$$

So as $T \to \infty$, we have $P(1,\infty) = \infty$ by the limit divergence test. Also,

$$R(1,T) = \int_1^T r(t)\Delta t = \sum_{t\in[1,T)_{2^{\mathbb{N}_0}}} \frac{1}{2t^{\frac{13}{5}}} \cdot t.$$

As $T \to \infty$, we have

$$\sum_{n=0}^{\infty} \left(\frac{1}{2\cdot 2^n}\right)^{\frac{8}{5}} < \infty$$

by the geometric series, i.e., $R(1,\infty) < \infty$. Finally, we have to show $Y_2 < \infty$.

$$\int_1^T r(t)g\left(k\int_1^t p(s)\Delta s\right)\Delta t = \int_1^T \frac{1}{2t^{\frac{13}{5}}} \left(\sum_{s\in[1,t)_{2^{\mathbb{N}_0}}} \left(\frac{s}{2s-1}\right)^{\frac{1}{61}} \cdot s\right)^{\frac{3}{5}} \Delta t$$

$$\leqslant \int_1^T \frac{1}{2t^{\frac{13}{5}}} \left(\sum_{s\in[1,t)_{2^{\mathbb{N}_0}}} s^{\frac{62}{61}}\right)^{\frac{3}{5}} \Delta t \leqslant \int_1^T \frac{1}{2t^{\frac{13}{5}}} \cdot t^{\frac{62}{105}}\Delta t = \sum_{t\in[1,T)_{2^{\mathbb{N}_0}}} \frac{1}{t^{\frac{208}{105}}}.$$

So as $t \to \infty$, we have

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^n}\right)^{\frac{208}{105}} < \infty$$

by the ratio test. Therefore, $Y_2 < \infty$ by the comparison test. One can also show that $(t, 2-\frac{1}{t})$ is a solution of

$$\begin{cases} \Delta_q x(t) = \left(\frac{t}{2t-1}\right)^{\frac{1}{61}} (y(t))^{\frac{1}{61}} \\ \Delta_q y(t) = \frac{1}{2t^{\frac{13}{5}}} (x(t))^{\frac{3}{5}} \end{cases}$$

such that $x(t) \to \infty$ and $y(t) \to 2$ as $t \to \infty$, i.e., $M^+_{\infty,B} \neq \emptyset$ by Theorem 4. Here Δ_q is known as a $q-$ derivative and defined as $\Delta_q h(t) = \frac{h(\sigma(t))-h(t)}{\mu(t)}$, where $\mu(t) = t$ is the graininess function and $\sigma(t) = 2t$ is the forward jump operator, see [2].

EXAMPLE 2. Let $\mathbb{T} = \{\frac{n}{2} : \ n \in \mathbb{N}_0\}$, $l = 1$, $f(z) = z^{\frac{1}{3}}$, $g(z) = z^{\frac{1}{5}}$, $p(t) = \frac{\sqrt{2}(\sqrt{2}-1)}{2^{\frac{2t}{3}}(3\cdot 2^t-1)^{\frac{1}{3}}}$, $r(t) = \frac{\sqrt{2}(\sqrt{2}-1)}{2^{\frac{4t}{5}}(2\cdot 2^t-1)^{\frac{1}{5}}}$, and $t = \frac{n}{2}$ in system (1). We show that there exists a nonoscillatory solution in $M^+_{B,B}$. So by Theorem 5, we need to show $P(t_0,\infty) < \infty$ and $R(t_0,\infty) < \infty$ and $Y_1 < \infty$. Indeed,

$$\int_0^T p(t)\Delta t = \sum_{t\in[0,T)_{\mathbb{T}}} \frac{\sqrt{2}(\sqrt{2}-1)}{2^{\frac{2t}{3}}(3\cdot 2^t-1)^{\frac{1}{3}}}\cdot\frac{1}{2} \leqslant \sum_{t\in[0,T)_{\mathbb{T}}} \frac{1}{2^{\frac{2t}{3}}}.$$

So as $T \to \infty$, we have

$$\sum_{n=0}^{\infty} \frac{1}{2^{\frac{n}{3}}} < \infty$$

by the geometric series, i.e., $P(t_0,\infty) < \infty$. Also

$$\int_0^T r(t)\Delta t = \sum_{t\in[0,T)_{\mathbb{T}}} \frac{\sqrt{2}(\sqrt{2}-1)}{2^{\frac{4t}{5}}(2\cdot 2^t-1)^{\frac{1}{5}}}\cdot\frac{1}{2} \leqslant \sum_{t\in[0,T)_{\mathbb{T}}} \frac{1}{2^{\frac{4t}{5}}}.$$

Hence, we have

$$\sum_{n=0}^{\infty} \frac{1}{2^{\frac{2n}{5}}} < \infty$$

as $T \to \infty$. It is also easy to show that $Y_1 < \infty$ if $P(t_0,\infty) < \infty$ and $R(t_0,\infty) < \infty$. It can be confirmed that $(2-\frac{1}{2^t}, 3-\frac{1}{2^t})$ is a nonoscillatory solution of

$$\begin{cases} x^{\Delta}(t) = \frac{\sqrt{2}(\sqrt{2}-1)}{2^{\frac{2t}{3}}(3\cdot 2^t-1)^{\frac{1}{3}}}(y(t))^{\frac{1}{3}} \\ y^{\Delta}(t) = \frac{\sqrt{2}(\sqrt{2}-1)}{2^{\frac{4t}{5}}(2\cdot 2^t-1)^{\frac{1}{5}}}(x(t))^{\frac{1}{5}} \end{cases}$$

such that $x(t) \to 2$ and $y(t) \to 3$ as $t \to \infty$, i.e., $M^+_{B,B} \neq \emptyset$ by Theorem 5.

EXAMPLE 3. Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $l = 1$, $f(z) = z^{\frac{1}{7}}$, $g(z) = z$, $p(t) = \frac{1}{4t^2(1+t)^{\frac{1}{7}}}$, $r(t) = \frac{2t}{4t-1}$, and $t = 2^n$ in system (1). We show that there exists a nonoscillatory solution in $M^+_{B,\infty}$. So we first need to confirm $P(t_0,\infty) < \infty$ and $R(t_0,\infty) = \infty$.

$$P(1,T) = \int_1^T p(t)\Delta t = \sum_{t\in[1,T)_{2^{\mathbb{N}_0}}} \frac{1}{4t^2(1+t)^{\frac{1}{7}}}\cdot t.$$

Similar to Example 1 and 2, we have that $P(1,\infty) < \infty$ as $T \to \infty$. It can also be shown that $R(1,\infty) = \infty$ and $Y_1 < \infty$ by using the similar idea in Example 1 and 2. It can be shown that $(2 - \frac{1}{2t}, t+1)$ is a nonoscillatory solution of

$$\begin{cases} \Delta_q x(t) = \frac{1}{4t^2(1+t)^{\frac{1}{7}}}(y(t))^{\frac{1}{7}} \\ \Delta_q y(t) = \frac{2t}{4t-1}x(t) \end{cases}$$

such that $x(t) \to 2$ and $y(t) \to \infty$ as $t \to \infty$, i.e., $M^+_{B,\infty} \neq \emptyset$ by Theorem 6.

3. Open problems and conclusion

In Section 1, we give the classification schemes for nonoscillatory solutions of system (1) by using $P(t_0,\infty)$, $R(t_0,\infty)$, Y_1, Y_2 and fixed point theorems. However, we cannot give the sufficient conditions for the existence of such solutions in $M^+_{\infty,\infty}$. It still remains as an open problem. Also note that if $R(t_0,\infty) < \infty$ and $R(t_0,\infty) < \infty$, then $M^+_{\infty,\infty} = \emptyset$. Another open problem for system (1) is to obtain the nonoscillation criteria in M^-. However, we might need extra assumptions such as the oddness of f or g in some cases. We can also consider system (2) for $r < 0$ and look for the nonoscillation criteria. By intuition, it may not be easy to obtain the (non)existence of nonoscillatory solutions of system (2) because of difficulty on the fixed point theorems and the chain rule on time scales.

Finally, we finish the paper with a summary of our main results in the following tables:

Table 1: *Existence for* (1) *in* M^+

$M^+_{\infty,B}$	$\neq \emptyset$	$P(t_0,\infty) = \infty$ and $R(t_0,\infty) < \infty$	$Y_2 < \infty$
$M^+_{B,B}$	$\neq \emptyset$	$P(t_0,\infty) < \infty$ and $R(t_0,\infty) < \infty$	$Y_1 < \infty$
$M^+_{B,\infty}$	$\neq \emptyset$	$P(t_0,\infty) < \infty$ and $R(t_0,\infty) = \infty$	$Y_1 < \infty$

Table 2: *Nonexistence for* (1) *in* M^+

$M^+_{\infty,B}$	$= \emptyset$	$P(t_0,\infty) = \infty$ and $R(t_0,\infty) < \infty$	$Y_2 = \infty$
$M^+_{B,B}$	$= \emptyset$	$P(t_0,\infty) < \infty$ and $R(t_0,\infty) < \infty$	$Y_1 = \infty$
$M^+_{B,\infty}$	$= \emptyset$	$P(t_0,\infty) < \infty$ and $R(t_0,\infty) = \infty$	$Y_1 = \infty$

REFERENCES

[1] D. R. ANDERSON, *Oscillation and Nonoscillation Criteria for Two-dimensional Time-Scale Systems of First Order Nonlinear Dynamic Equations*, Electron. J. Differential Equations, vol. 2009 (2009), no. 24, pp. 1–13.

[2] M. BOHNER AND A. PETERSON, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[3] M. BOHNER AND A. PETERSON, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[4] M. CECCHI, Z. DOŠLÁ, M. MARINI AND I. VRKOČ, *Integral Conditions for Nonoscillation of Second Order Nonlinear Differential Equations*, Nonlinear Anal. **64** (2006) 1278–1289.

[5] P. G. CIARLET, *Linear and Nonlinear Functional Analysis with Applications*, Siam, 2013.

[6] B. KNASTER, *Un théorème sur les fonctions d'ensembles*, Ann. Soc. Polon. Math. **6** (1928) 133–134.

[7] W. T. LI, *Classification Schemes for Positive Solutions of Nonlinear Differential Systems*, Math. Comput. Modelling **36** (2002) 411–418.

[8] Ö. ÖZTÜRK, E. AKIN, *Classification of Nonoscillatory Solutions of Nonlinear Dynamic Equations on Time Scales*, Dynam. Systems Appl., to appear, 2015.

[9] Ö. ÖZTÜRK, E. AKIN, İ. U. TIRYAKI, *On Nonoscillatory Solutions of Emden-Fowler Dynamic Systems on Time Scales*, Filomat, to appear, 2015.

[10] Ö. ÖZTÜRK, E. AKIN, *Nonoscillation Criteria for Two Dimensional Time Scale Systems*, Nonauton. Dyn. Syst. 2016, 3: 1–13.

[11] Ö. ÖZTÜRK, E. AKIN, *On Nonoscillatory Solutions of Two Dimensional Nonlinear Delay Dynamical Systems*, Opuscula Math., vol. 36, no. 5 (2016).

[12] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications – I: Fixed Point Theorems*, Springer Verlag New York Inc, 1986.

(Received May 27, 2016)

Özkan Öztürk
Missouri University of Science and Technology
314 Rolla Building, Rolla, MO, USA, 65409 – 0020
e-mail: oo976@mst.edu